Solve the following 6 problems.

1. Prove that if series \( \sum_{n=1}^{\infty} a_n x^n \) converges for all \( x \) such that \( |x| < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n} \) converges as well if \( |x| < 1 \).

2. Let \( X \) be a nonempty set and \( d \) be a metric on \( X \). Prove the standard theorem that the set of all limit points of \( X \) is closed.

3. Let \( X \) be a nonempty set and \( d \) be a metric on \( X \). We say that \( K \subset X \) is **sequentially compact** if for every sequence \( \{x_n\} \subset K \) there exists a subsequence \( \{a_{n_k}\} \) that converges to a point \( x \in K \). For a fixed \( \epsilon > 0 \), we call \( \{x_\alpha\}_{\alpha \in \mathcal{A}} \subset X \) an \( \epsilon \)-net of \( K \subset X \) if the family of open balls \( \{B_\epsilon(x_\alpha)\}_{\alpha \in \mathcal{A}} \) is an open cover of \( K \). We say that \( K \subset X \) is totally bounded if there exists a finite \( \epsilon \)-net for every \( \epsilon > 0 \). Use these definitions to prove the standard theorem that a nonempty sequentially compact subset of a metric space is complete and totally bounded.

4. Let \( X \) be a nonempty set and \( d \) be a metric on \( X \). Suppose \( f \) is a continuous function on \( A \subset X \) to \( \mathbb{R}^n \) for some \( n \in \mathbb{N} \). Using only the definitions of a set being compact and a function being uniformly continuous, prove that if \( A \) is compact, then \( f \) is uniformly continuous, and provide a counterexample to the converse.

5. Let \( a < b \) be real numbers and \( f : [a, b] \rightarrow \mathbb{R} \). For a partition \( P = \{a = x_0 < x_1 < \cdots < x_n = b\} \) of \([a, b]\), the upper and lower Darboux sums of \( f \) on \( P \) are defined as

\[
U(f, P) = \sum_{i=1}^{n} \left( \sup_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}),
\]

and

\[
L(f, P) = \sum_{i=1}^{n} \left( \inf_{y \in [x_{i-1}, x_i]} f(y) \right) (x_i - x_{i-1}),
\]

respectively. We say that \( f \) is Riemann integrable on \([a, b]\) if for every \( \epsilon > 0 \) there exists a partition \( P \) of \([a, b]\) such that \( U(f, P) - L(f, P) < \epsilon \).

Suppose that \( f : [a, b] \rightarrow \mathbb{R} \) is bounded. Using the above definitions, prove that if \( f \) is Riemann integrable, then \( f^2 \) is Riemann integrable, and provide a counterexample to the converse.

*Hint:* You may find it useful to exploit the fact that for any set \( A \), and any real-valued function \( f \) defined on \( A \) that \( \sup_{x \in A} f(x) - \inf_{y \in A} f(y) = \sup_{x,y \in A} |f(x) - f(y)| \).

6. Let \( C^1([a, b]) \) denote the space of real-valued continuously differentiable functions on \([a, b]\) where \( a < b \) are real numbers. Define the metric \( d \) on \( C^1([a, b]) \) as follows (where \( f, g \in C^1([a, b]) \))

\[
d(f, g) = \sup_{[a,b]} |f(x) - g(x)| + \sup_{[a,b]} |f'(x) - g'(x)|.
\]

Suppose \( \{f_n\} \subset (C^1([a, b]), d) \) is a bounded sequence. Prove that if \( \{f'_n\} \) is equicontinuous, then there exists a subsequence of \( \{f_n\} \) that converges in \( C^1([a, b], d) \).