PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
JANUARY 19, 2018

Name: ________________________________

• The examination consists of 6 problems. Do problems 1 to 4, and two out of 5, 6, and 7. If you submit all three, only 5 and 6 will be graded.
• Each problem is worth 20 points. Numbered parts of a problem have equal weight.
• Justify your solutions: cite theorems that you use, provide counter-examples, give explanations.
• Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
• Begin solution to every problem on a new page; write only on one side of a sheet; number all pages throughout; just in case, write your name on every page.
• Do not submit scratch paper.
• Ask the proctor if you have any questions.

   Good luck!

1. _______

2. _______

3. _______

4. _______

5. _______

6. _______

7. _______

Total ______

Examination committee: Troy Butler, Jan Mandel (chair), Florian Pfender
(1) Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and \(K \subset X\) be compact. Prove the standard result that if \(f : X \to Y\) is continuous, then \(f(K) \subset Y\) is compact.
Let \((X, d)\) be a metric space, \(A \subset X\) nonempty, and for any \(x \in X\), define the distance from \(x\) to \(A\) as
\[
dist(x, A) = \inf\{d(x, a) : a \in A\}.
\]
We say that \(y\) is a limit point of \(A\) if for all \(r > 0\) there exists \(a \in A\) such that \(d(a, y) < r\). Using this definition of a limit point, show that \(y\) is a limit point of \(A\) if and only if \(\text{dist}(y, A) = 0\).
(3) Let $a < b$ be real numbers and $(f_n)$ be a sequence of contraction maps such that $f_n : [a, b] \to [a, b]$ for all $n$, prove the following:

(a) There exists a uniformly convergent subsequence $(f_{n_k})$.

(b) If $f$ denotes the limit of the uniformly convergent subsequence, then there exists $x \in [a, b]$ such that $f(x) = x$. 
(4) Let \((X, d)\) be a compact metric space and \((f_n)\) a sequence of continuous real-valued functions defined on \(X\) that converge pointwise to a continuous function \(f\). Prove the standard result: if \(f_n(x) \geq f_{n+1}(x)\) for all \(x \in X\) and \(n \in \mathbb{N}\), then \((f_n)\) converge uniformly.
(5) Let $L < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable with the property that $\sup_{x \in \mathbb{R}} f'(x) < L$. Prove that there exists a fixed point for $f$, i.e., that there exists $x \in \mathbb{R}$ such that $f(x) = x$. Hint: Consider the function $g(x) = x - f(x)$. 
(6) Define $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that derivatives of $f$ of all orders exist at 0, and $f^{(n)}(0) = 0$. 
(7) Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ Riemann integrable. Prove that for all $\varepsilon > 0$ there exists $g \in C([a, b])$ such that

$$\int_a^b |f - g| \, dx < \varepsilon.$$ 

Hint: Define $g$ piecewise.