1. Let \((X, d)\) be a metric space.

(a) Prove or find a counterexample: If \((x_n)\) is a Cauchy sequence in \((X, d)\), then \((x_n)\) converges.
(b) Prove that if \((x_n)\) and \((y_n)\) are both Cauchy sequences in \((X, d)\), then the sequence \((d(x_n, y_n))\) converges.

2. Let \((a_n), (b_n)\) be sequences in \(\mathbb{R}\) and suppose \(\lim_{n \to \infty} a_n = a \in \mathbb{R}\).

(a) Prove that if \(a > 0\), then \(\liminf_{n \to \infty} a_nb_n = a \liminf_{n \to \infty} b_n\).

\[ \text{Hint: There is no assumption that } \liminf_{n \to \infty} b_n \text{ is finite. You can do the finite and infinite cases separately, or try to do them together.} \]

\[ \text{Hint: You must clearly indicate where the assumption } a \neq 0 \text{ is used in the proof.} \]

(b) Provide a counterexample when the statement fails with \(a = 0\) and \(\liminf_{n \to \infty} b_n \in \mathbb{R}\).

3. Prove that \(f_n : E \to \mathbb{R}\) and \((f_n)\) is uniformly convergent on every finite or countable subset of \(E\), then \((f_n)\) is uniformly convergent on \(E\).

4. Assume that \(a_n \in \mathbb{R}\) for all \(n\), and \(\sum_{n=1}^{\infty} \frac{a_n}{x^2}\) converges for \(x = x_0 \in \mathbb{R}\). Show that then the series converges for all \(x > x_0\). (Be careful: there is no assumption on the signs of the \(a_n\).)

5. Let \(\sum_{n \geq 0} u_n\) be a convergent series with real nonnegative terms, \(u_n \geq 0\). For all \(n \in \mathbb{N}\), we define \(v_n = \sup_{p \geq n} u_p\). Does it follow that the series \(\sum_{n \geq 0} v_n\) converges?

6. Suppose that \(A \subset [0, 1]\) is a countable set with only a single limit point \(x_0 \in (0, 1)\). Define \(f : [0, 1] \to \mathbb{R}\) by

\[
 f(x) = \begin{cases} 
 1 & \text{if } x \in A \\
 0 & \text{otherwise} 
\end{cases}
\]

Using the definition of Riemann integral, find if the Riemann integral \(\int f(x)\,dx\) exists, and find its value if it does.