August 2020 Analysis Prelim Solutions

September 9, 2020

Section 1: do all four problems

1. (20 points) Let B be a set in a metric space (X, d). Let B' be the set of all limit points of B. State the definition of a limit point and prove that B' is closed.

   Solution. This is a standard result. The precise wording of the proof depends on the definition you use for limit point. Here we will use Rudin's definition, which is

   $$x \in B' \text{ if } \forall \epsilon > 0 : N_\epsilon(x) \cap B - \{x\} \neq \emptyset.$$

   Suppose B' is not closed. Then there is a sequence $(x_n) \subset B'$ with $x_n \to x \notin B'$. Since $x \notin B'$, there exists $\epsilon > 0$ such that

   $$N_\epsilon(x) \cap B - \{x\} = \emptyset.$$

   Since $x_n \to x$, there exists x_k such that $d(x_k, x) < \epsilon/2$. Let $\epsilon' = d(x_k, x)$, so $x \notin N_{\epsilon'}(x_k)$. From the triangle inequality, if $z \in N_{\epsilon'}(x_k)$, then

   $$d(z, x) \leq d(z, x_k) + d(x_k, x) < \epsilon' + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$$

   thus $N_{\epsilon'}(x_k) \subset N_\epsilon(x)$, so

   $$N_{\epsilon'}(x_k) \cap B - \{x_k\} = \emptyset,$$

   which contradicts the assumption that $x_k \in B'$.

2. (20 points) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be uniformly continuous. Let (x_n) be a Cauchy sequence in X. Prove that $(f(x_n))$ is a Cauchy sequence in Y.

   Solution. This is also a standard result. Let $\epsilon > 0$. Since $f : X \to Y$ is uniformly continuous, there is a $\delta > 0$ such that if $d_X(a, b) < \delta$ then $d_Y(f(a), f(b)) < \epsilon$. Since (x_n) is a Cauchy sequence in (X, d_X), we can choose N such that

   $$n, m \geq N \Rightarrow d_X(x_n, x_m) < \delta.$$

   So if $n, m \geq N$, then $d_Y(f(x_n), f(x_m)) < \epsilon$. Since ϵ was arbitrary, $(f(x_n))$ is Cauchy in (Y, d_Y).

3. (20 points) Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is differentiable at $x = 0$ and discontinuous at every other $x \in \mathbb{R}$. Prove that your example works.

Solution. A simple example is

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

where $\mathbb{Q}$ is the set of rational numbers. Since $|f(x)| \le x^2$,

$$\lim_{x\to 0} \left| \frac{f(x) - f(0)}{x - 0} \right| \le \lim_{x\to 0} \left| \frac{x^2}{x} \right| = 0,$$

f is differentiable at 0 with $f'(0) = 0$. Suppose $x \neq 0$ and $x \in \mathbb{Q}$. Then there is a sequence of irrational numbers $z_n \to x$. But $f(z_n) = 0$, so $f(z_n) \not\to f(x) = x^2 \neq 0$. So f is not continuous at x. Likewise, if x is irrational then there is a sequence of rational numbers (y_n) that converges to x. But $f(y_n) = y_n^2 \to x^2 \neq 0 = f(x)$ so f is not continuous at x. Since every nonzero real number is either rational or irrational, f is discontinuous at every $x \neq 0$.

4. (20 points) For all $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a function with at most one discontinuity. Further assume that $f_n \to f$ uniformly for some function $f : \mathbb{R} \to \mathbb{R}$. Prove that f has at most one discontinuity.

   Solution. Suppose f is discontinuous at x_1 and x_2. Then there exists $\delta > 0$ such that

$$\forall \epsilon > 0 \exists y_1, y_2 \in \mathbb{R} : |x_i - y_i| < \epsilon,\ |f(x_i) - f(y_i)| > \delta,\ i = 1, 2.$$

   Since $f_n \to f$ uniformly, we can choose N so that if $n \ge N$, then for all x, $|f_n(x) - f(x)| < \delta/3$. Let $n \ge N$. Let $\epsilon > 0$, and choose y_1 and y_2 from above. Then for $i = 1, 2$,

$$\underbrace{|f(x_i) - f(y_i)|}_{>\delta} \le \underbrace{|f(x_i) - f_n(x_i)|}_{<\delta/3} + |f_n(x_i) - f_n(y_i)| + \underbrace{|f_n(y_i) - f(y_i)|}_{<\delta/3}$$

   thus

$$|f_n(x_i) - f_n(y_i)| > \delta - \delta/3 - \delta/3 = \delta/3.$$

   We have proved that for every $\epsilon > 0$, there exist y_1 and y_2 so that $|x_i - y_1| < \epsilon$ and $|f_n(x_i) - f_n(y_i)| > \delta/3$, $i = 1, 2$. But then f_n is discontinuous at x_1 and x_2, which contradicts the assumption that it has at most one discontinuity.

Section 2: do two of the following four problems

5. Let $f : [0, 1] \to \mathbb{R}$ have the following property: For every $x \in [0, 1]$ and $\epsilon > 0$, there exists $\delta > 0$, such that for every $y \in [0, 1]$ with $|x - y| < \delta$, we have $f(y) < f(x) + \epsilon$.

   (a) (10 points) Prove that f is bounded above, and

   (b) (10 points) f attains its maximum (i.e., there exists $z \in [0, 1]$ such that $f(x) \le f(z)$ for all $x \in [0, 1]$).

   Solution.

   (a) Fix $\epsilon > 0$. For every x, choose $\delta_x > 0$ such that

$$|x - y| < \delta_x \Rightarrow f(y) < f(x) + \epsilon.$$

Define $O_x = (x - \delta_x, x + \delta_x)$. Then $\{O_x : x \in [0,1]\}$ is an open cover of $[0,1]$. Since $[0,1]$ is compact, there exist $\{x_1, x_2, \ldots, x_n\}$ such that

$$[0,1] \subset \bigcup_{i=1}^{n} O_{x_i}.$$

Thus, every $x \in [0,1]$ is in at least one of the O_{x_i}, $i = 1, \ldots, n$, and, consequently,

$$f(x) < \max\{f(x_1), f(x_2), \ldots, f(x_n)\} + \epsilon,$$

so f is bounded above.

(b) Denote $y = \sup_{x\in[0,1]} f(x)$. From part (a), $y < \infty$ and from the properties of supremum, there exists a sequence $(x_n) \subset [0,1]$ such that $f(x_n) \to y$. Since $[0,1]$ is compact, there is a subsequence $x_{n_i} \to z$ for some $z \in [0,1]$. Suppose $f(z) < y$. Then

$$f(z) < y - \epsilon, \text{ with } \epsilon = (y - f(z))/2 > 0.$$

From the assumption on f, we can choose $\delta > 0$ so that

$$|x - z| < \delta \Rightarrow f(x) < f(z) + \epsilon/2 < y - \epsilon + \epsilon/2 = y - \epsilon/2.$$

But $f(x_{n_i}) \to y$, so there is an m such that

$$i > m \Rightarrow f(x_{n_i}) > y - \epsilon/2.$$

Since $x_{n_i} \to z$, there exists $i > m$ such that $|x_{n_i} - z| < \delta$. But this implies $f(x_{n_i}) < y - \epsilon/2$, which is a contradiction. Thus $f(z) \geq y$. But since $f(z) \leq y$ by the definition of y, we have $f(z) = y$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with

$$\lim_{x\to\infty} \left(f(x) + \int_0^x f(t)\, dt \right) = 0.$$

(a) (5 points) Suppose $f(x) \not\to 0$. Prove that there exists $\epsilon > 0$ such that for any $K > 0$, there exist $x > K$ and $y > x$ such that $|f(x)| < \epsilon/2$ and $|f(y)| > \epsilon$.

(b) (15 points) Use (a) to prove that $\lim_{x\to\infty} f(x) = 0$.

Solution.

(a) We are given that

$$g(x) \equiv f(x) + \int_0^x f(t)dt \to 0 \text{ as } x \to \infty. \tag{1}$$

Suppose that $f(x) \not\to 0$ as $x \to \infty$, which is the same as

$$\exists \epsilon > 0 \forall K > 0\ \exists z > K : |f(z)| > \epsilon. \tag{2}$$

Fix ϵ satisfying (2). Let $K > 0$. Take $z > K$ so that $f(z) > \epsilon$. (The argument is identical if $f(z) < -\epsilon$.) Suppose that $f(x) \geq \epsilon/2$ for all $x > z$. Then, for any $x > z$,

$$g(x) = f(x) + \int_0^x f(t)dt \geq \epsilon/2 + \int_0^z f(t)dt + (x - z)\epsilon/2,$$

and since $\int_0^z f(t)dt \in \mathbb{R}$, it follows that $g(x) \to \infty$, which contradicts (1). Thus, there exists $x > K$ such that $|f(x)| < \epsilon/2$. Using (2) with x in place of K, we get that there exists $y > x$ such that $|f(y)| > \epsilon$.

(b) Suppose $f(x) \not\to 0$. Take $\epsilon > 0$ as in part (a). Since $g(x) \to 0$ as $x \to \infty$, there exists $K > 0$ such that

$$x > K \Rightarrow |g(x)| < \epsilon/4 \tag{3}$$

From part (a), there exist x, y such that

$$x > K, \quad y > x, \quad |f(x)| < \epsilon/2, \quad |f(y)| > \epsilon.$$

Assume that $f(y) > \epsilon$ since the argument is identical if $f(y) < -\epsilon$. Define

$$u = \sup\{v \in (x, y) \ : \ f(v) < \epsilon/2\}.$$

Since f is continuous, $f(u) = \epsilon/2$. By the construction of u, for all $t \in (u, y)$, $f(t) \geq \epsilon/2$. Thus, using (3)

$$\begin{aligned}
\epsilon/4 > g(y) &= f(y) + \int_0^y f(t)dt \\
&= f(u) + (f(y) - f(u)) + \int_0^u f(t)dt + \int_u^y f(t)dt \\
&= \underbrace{g(u)}_{>-\epsilon/4} + (\underbrace{f(y)}_{>\epsilon} - \underbrace{f(u)}_{=\epsilon/2}) + \int_u^y \underbrace{f(t)}_{\geq\epsilon/2} dt \\
&> -\epsilon/4 + \epsilon - \epsilon/2 + (y-u)\epsilon/2 \geq \epsilon/4
\end{aligned}$$

which is a contradiction.

7. (20 points) Let M be a metric space. Prove that if every nested decreasing sequence (X_n) of closed nonempty subsets $M \supseteq X_1 \supseteq X_2 \supseteq \ldots$ has $\bigcap X_n \neq \emptyset$, then M is compact. Hint: You can use without proof that sequentially compact space is compact.

   Solution. Suppose that M is not compact, then M is not sequentially compact, i.e., there exists a sequence $(x_n) \subset M$ which has no convergent subsequence. Denote $X_n = \{x_n, x_{n+1}, \ldots\}$. Then X_1 has no limit points, so every subset of X_1 is closed, in particular every X_n is closed. Since $X_n \neq \emptyset$, X_n is closed, and (X_n) is a decreasing sequence of sets, by assumption $\cap_{n=1}^{\infty} X_n \neq \emptyset$. But if $x \in \cap_{n=1}^{\infty} X_n$, then x equals to inifinitely many x_{n_i}; otherwise, the set $\{i : x_i = x\}$ is finite,

$$x \notin \cap_{\max\{i : x_i = x\}} X_n \subset \cap_{n=1}^{\infty} X_n,$$

   a contradiction with $x \in \cap_{n=1}^{\infty} X_n$. But (x_{n_i}) is a constant sequence and thus a convergent subsequence of (x_n), which contradicts the assumption that (x_n) has no convergent subsequence. .

8. Let M be a compact metric space, let $f_n : M \to \mathbb{R}$ be a sequence of continuous functions, such that for all $x \in M$, $(f_n(x))$ is monotonically decreasing with limit 0. Prove that f_n converges uniformly to the zero function. Here is one path to a proof, but you can try another, e.g., using the converse of the statement of problem 7.

   (a) (5 points) Argue that each f_n attains its maximum at some $x_n \in M$.

   (b) (5 points) Prove that $(f_n(x_n))$ is a monotonically decreasing sequence.

   (c) (10 points) Argue from (a) and (b) that $f_n \to 0$ uniformly.

Solution.

We will give two proofs. The first uses parts (a),(b),(c) suggested on the exam, and (d) is an alternate proof.

(a) Continuous function on a compact metric space attains its maximum (e.g., Rudin Theorem 4.16), thus there exists $x_n \in M$ such that $f_n(x_n) = \max_{x \in M} f_n(x)$.

(b) We have

$$f_{n+1}(x_{n+1}) \le f_n(x_{n+1}) \le f_n(x_n).$$

The first inequality follows since for each x, $(f_n(x))$ is a decreasing sequence. The second inequality follows since x_n is where f_n is maximized.

(c) Since $f_n(x_n)$ is decreasing and nonnegative, it converges to some limit, $L \ge 0$. Since M is compact, there is a subsequence (x_{n_i}) that converges to some $x \in M$. If $n_k > N$ then $f_N(x_{n_k}) \ge f_{n_k}(x_{n_k})$ since since the sequence $(f_n(x_{n_k}))$ is monotonically decreasing by assumption, thus for any N,

$$f_N(x) = \lim_{k\to\infty} f_N(x_{n_k}) \ge \lim_{k\to\infty} f_{n_k}(x_{n_k}) = L,$$

since $n_k > N$ for k large enough. Similarly, since for a fixed n, $f_n(x_{n_k}) \ge f_{n_k}(x_{n_k})$ for large enough $n_k > N$, we have

$$0 = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{k\to\infty} f_n(x_{n_k}) \ge \lim_{n\to\infty} \lim_{k\to\infty} f_{n_k}(x_{n_k}) = \lim_{n\to\infty} L = L.$$

Thus, $L = 0$, and $\lim_{n\to\infty} f_n(x_n) = 0$. To show this implies uniform convergence, let $\epsilon > 0$. Since $\lim_{n\to\infty} f_n(x_n) = 0$, there exists N such that $f_n(x_n) < \epsilon$ when $n \ge N$. Then for any $n \ge N$ and $x \in M$, $f_n(x) \le f_n(x_n) < \epsilon$.

(d) As an alternate proof, fix $\epsilon > 0$ and define

$$X_n = \{x \in M \;:\; f_n(x) \ge \epsilon\}.$$

Since f_n is continuous, X_n is closed, thus compact. Furthermore, X_n is a decreasing sequence of sets since $f_n(x)$ is decreasing for every x. It is known that if each X_n is nonempty then $\cap_{n=1}^{\infty} X_n \ne \emptyset$ (the Cantor intersection theorem, Rudin 2.36). But, if $x \in \cap_{n=1}^{\infty} X_n$ then $f_n(x) \ge \epsilon$ for all n, which implies that $f_n(x) \not\to 0$ which is a contradiction. Thus there is N such that $X_N = \emptyset$, hence for all $n \ge N$, $X_n \subset X_N = \emptyset$, that is, $f_n(x) < \epsilon$ by the definition of X_n. Since ϵ was arbitrary, $f_n \to 0$ uniformly.