University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 9, 2017

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. You are asked to submit solutions to *6 problems*. If you submit solutions to more than six problems, you must indicate which problems to grade. If you do not indicate which problems to grade, only the first six solutions will contribute to your grade. Your final score will be out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce a complete proof.
- Parts of a multipart question are not necessarily worth the same number of points.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

1.	____________	5.	____________
2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________
		Total	____________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Steffen Borgwardt, Varis Carey, and Stephen Hartke (Chair).

Problem 1

Consider the map $\varphi : \mathbb{R}^4 \to \mathbb{R}^{2\times 2}$ where

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mapsto \begin{pmatrix} a+b-c & c-d \\ 2a+c & a-b+d \end{pmatrix}$$

a) Is φ bijective? Prove your claim.

b) Compute $(a, b, c, d)^T$ such that $\varphi((a, b, c, d)^T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or decide that this is not possible.

Problem 2

Let $x_1, \ldots, x_n \in \mathbb{C}$ with $n \geq 2$. Then

$$V(x_1, \ldots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

is called a *Vandermonde Matrix*. Prove that

$$\det(V(x_1, \ldots, x_n)) = \prod_{1 \leq i < j \leq n} (x_j - x_i).$$

Problem 3

Let $V = \mathbb{R}[x]_{\leq 2} = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ be the space of polynomials of degree ≤ 2. Let $u_1 = 1$, $u_2 = x$, $u_3 = x^2$ and $b_1 = 1$, $b_2 = x+1$, $b_3 = x^2 + x + 1$. Then $E = \{u_1, u_2, u_3\}$ and $B = \{b_1, b_2, b_3\}$ are bases of V. Further, let $C = \{c_1, c_2, c_3\}$ be a basis that is not explicitly known. The only available information is the basis transformation matrix

$$S_{E,C} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 3 \end{pmatrix}.$$

Notation. $S_{E,C}$ is the representation matrix of the identity $id\colon \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto x$, where the coordinates of x with respect to E are given before the mapping, and the coordinates with respect to C are given after the mapping.

a) Compute the basis transformation matrices $S_{E,B}$ and $S_{B,E}$.

b) Let $y := (1, 2, -1)^T \in \mathbb{R}^3$ be the coordinate vector of a vector $v \in V$ with respect to basis B. What is the coordinate vector of v with respect to E? Further, compute v explicitly.

c) What is the coordinate vector of $w := 2 - 3x + x^2 \in V$ with respect to C? Further, write w as a linear combination of the elements of C.

d) Compute the basis C.

Problem 4

Two $n \times n$ real matrices A and B are called *simultaneously diagonalizable* if there is an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $S^{-1}AS$ and $S^{-1}BS$ both are diagonal matrices. Let A and B be two $n \times n$ real matrices. Prove:

a) If A and B are simultaneously diagonalizable, then $AB = BA$.

b) If $AB = BA$ and if A has n different eigenvalues, then A and B are simultaneously diagonalizable.

Problem 5

Let $A \in \mathbb{R}^{3\times 3}$ be an unknown matrix, and let

$$v_1 := \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^3.$$

Further let $S := (v_1|v_2|v_3) \in \mathbb{R}^{3\times 3}$ be the real 3×3 matrix with columns v_1, v_2, v_3. Finally, let

$$\ker(A + 2I_3) = \operatorname{span}\langle v_1 \rangle \qquad \text{and} \qquad \ker(A - I_3) = \operatorname{span}\langle v_2, v_3 \rangle,$$

where I_3 is the 3×3 identity matrix.

a) Prove that A is diagonalizable and give the characteristic polynomial χ_A in factorized form. Further, give all eigenvalues of A with their geometric and algebraic multiplicities. Finally, give the minimal polynomial m_A of A.

b) Compute the matrix A.

Problem 6

Suppose you are given eight 6×6 complex matrices whose cube is zero (i.e., $A^3 = 0$). Show that two of the matrices must be similar.

Problem 7

Let V be a real finite-dimensional inner-product space with proper subspaces U and W. Let P_U and P_W be the orthogonal projections onto U and W, respectively.

(a) Prove or give a counterexample: $P_U P_W = P_W P_U = P_{U \cap W}$.

(b) Prove that $\text{trace}(P_U) = \dim U$.

(c) Let $V = \mathbb{C}^n$, and let $\dim U$=1. Write down $\mathcal{M}(P_U)$ with respect to the standard basis for V.

Problem 8

Let T be an operator on a finite-dimensional complex valued vector space V. Suppose that there exists a $v \in V$ such that $\{v, Tv, T^2v, \dots T^{n-1}v\}$ is a basis for V.

(a) Prove that $\mathcal{M}(T)$ can be written with respect to this basis such that the matrix is zero below the principal subdiagonal (i.e., that the matrix is upper Hessenberg).

(b) Let T be diagonalizable. Prove that T is invertible.

(c) If T is not invertible, prove that the range of T is invariant under T.