University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
Friday, June 12, 2015

Name: ____

Exam Rules:

- This exam lasts 4 hours and consists of 6 problems worth 20 points each.
- Each problem will be graded, and your final score will count out of 120 points.
- You are not allowed to use your books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, ..., 2-1, 2-2, 2-3, ...).
- Read all problems carefully and write your solutions legibly using a **dark pencil or pen** "essay-style" (using full sentences) and in correct mathematical notation.
- Justify all your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce an independent proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

1.	________	4.	________
2.	________	5.	________
3.	________	6.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

For each of the following parts, determine all real 2-by-2 matrices A that satisfy the following:

(a) $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; (b) $AA^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. [10 points each]

Problem 2

Let A be a real symmetric n-by-n matrix. Show that the following three statements are equivalent. [20 points]

(a) All the eigenvalues of A are positive.

(b) For every nonzero $x \in \mathbb{R}^n$, one has $x^T A x > 0$.

(c) There exists an invertible matrix Q such that $A = QQ^T$.

Problem 3

(a) Let A be an m-by-n matrix. Prove that if the matrix $A^T A$ is invertible, then the matrix

$$I - A(A^T A)^{-1} A^T$$

is symmetric positive semidefinite. [10 points]

(b) In addition, let B be an m-by-p matrix. Prove that if $A^T A$ and $B^T B$ are invertible and if the ranges of A and B do not share a nontrivial subspace, then the matrix

$$B^T (I - A(A^T A)^{-1} A^T) B$$

is invertible. [10 points]

Problem 4

A square matrix N is called nilpotent if $N^m = 0$ for some positive integer m.

(a) Is the sum of two nilpotent matrices nilpotent? [5 pts]
If yes, prove it. If not, give a counterexample.

(b) Is the product of two nilpotent matrices nilpotent? [5 pts]
If yes, prove it. If not, give a counterexample.

(c) Prove that all eigenvalues of a nilpotent matrix are zero. [5 pts]

(d) Prove that the only nilpotent matrix that is diagonalizable is the zero matrix. [5 pts]

Problem 5

In this problem, you are asked to prove that two real symmetric matrices commute if and only if they are diagonalizable in a common orthonormal basis. We suggest the following path.

Let A and B be two real symmetric matrices and show each of the following. [5 points each]

(a) If A and B are diagonalizable in a common orthonormal basis, then A and B commute.

(b) If A and B commute, and if λ is an eigenvalue of A, then the eigenspace E_λ of A that is associated with the eigenvalue λ is invariant under B.

(c) If A and B commute, then A and B have at least one common eigenvector.

(d) If A and B commute, then A and B are diagonalizable in a common orthonormal basis.

Problem 6

Let V be a 4-dimensional vector space over $\mathbb{R}$, let $\mathcal{L}(V, V)$ be the set of all linear mappings from V to V, and let $T\colon V \to V$ be a linear operator with minimal polynomial $\mu_T(x) = x^2+1$. Determine, with a proof, the dimension of the following subspace:

$$U(T) := \{S \in \mathcal{L}(V, V) \mid ST = TS\}. \qquad \text{[20 points]}$$