University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
Friday, June 12, 2015

Name: ____

Exam Rules:

- This exam lasts 4 hours and consists of 6 problems worth 20 points each.
- Each problem will be graded, and your final score will count out of 120 points.
- You are not allowed to use your books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully and write your solutions legibly using a **dark pencil or pen** "essay-style" (using full sentences) and in correct mathematical notation.
- Justify all your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce an independent proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

1.	____________	4.	____________
2.	____________	5.	____________
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Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

For each of the following parts, determine all real 2-by-2 matrices A that satisfy the following:

(a) $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; (b) $AA^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. [10 points each]

Solution 1 ("brute force") Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ so that:

$$AA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}.$$

For (a), it follows that $a^2 + bc = 1$ and $d^2 + bc = -1$ so that $a^2 - d^2 = (a + d)(a - d) = 2$ and thus $a + d \neq 0$. Hence, from $b(a + d) = c(a + d) = 0$, we get $b = c = 0$ yielding $d^2 + bc = d^2 = -1$ which shows that such matrix cannot exist. Similarly, for (b), we get:

$$AA^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix},$$

and the same conclusion follows immediately because $c^2 + d^2 \geq 0 > -1$.

Solution 2 (using determinant) For (a), on the one hand,

$$\det(A^2) = \det(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = -1.$$

On the other hand,

$$\det(A^2) = (\det(A))^2.$$

Therefore

$$(\det(A))^2 = -1.$$

Since A is real, $\det(A)$ is real. No such A exists.
For (b), on the one hand,

$$\det(AA^T) = \det(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) = -1.$$

On the other hand,

$$\det(AA^T) = \det(A)\det(A^T) = (\det(A))^2.$$

Therefore

$$(\det(A))^2 = -1.$$

Since A is real, $\det(A)$ is real. No such A exists.

Solution 3 (using eigenvalues for (a) and a basic observation for (b)) For (a), if

$$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then one could observe that:

$$(A^2 - I)(A^2 + I) = 0,$$

so that the eigenvalues of A are among ± 1 and $\pm i$. Moreover, because A is real so that its complex eigenvalues come in complex conjugate pairs, it follows that if A has the eigenvalue i, then $-i$ is also an eigenvalue of A, and A^2 would have the eigenvalue -1 twice. (Ditto if $-i$ is an eigenvalue of A.) Clearly, such A^2 does not satisfy our equation which rules out that i and $-i$ are eigenvalues for A. Similarly, if A has eigenvalues 1 and -1, then A^2 would have the eigenvalue 1 twice and clearly, such A^2 again does not satisfy our equation. As a consequence, we can see that there is no real 2-by-2 matrix that satisfies (a). For any A in (b), we can observe that AA^T has to be symmetric positive semidefinite whereas the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is not. As a consequence, we see that there is no real 2-by-2 matrix that satisfies (b).

Problem 2

Let A be a real symmetric n-by-n matrix. Show that the following three statements are equivalent. [20 points]

(a) All the eigenvalues of A are positive.

(b) For every nonzero $x \in \mathbb{R}^n$, one has $x^T Ax > 0$.

(c) There exists an invertible matrix Q such that $A = QQ^T$.

Solution We understand that all these three statements relates symmetric positive definite matrices. Any one of the three statements above can be used as the definition for a symmetric positive definite matrix.

(c) $\Rightarrow$ (b) We assume that there exists an invertible n-by-n matrix Q such that $A = QQ^T$. Let x be a nonzero vector in $\mathbb{R}^n$, then:

$$x^T Ax = x^T QQ^T x = (Q^T x)^T (Q^T x) = \|Q^T x\|_2^2.$$

Since Q is invertible, Q^T is invertible, and since x is nonzero, $Q^T x$ is nonzero. Since $Q^T x$ is nonzero, $\left\|Q^T x\right\|_2^2 > 0$. So we proved that, for every nonzero $x \in \mathbb{R}^n$, one has

$$x^T Ax > 0.$$

(b) $\Rightarrow$ (a) We assume that, for every nonzero $x \in \mathbb{R}^n$, one has $x^T Ax > 0$. Let λ be an eigenvalue of A and v be an associated eigenvector such that $Av = \lambda v$ and $v \neq 0$. Now $v^T Av = \lambda v^T v$ but $v^T Av > 0$ by our assumption, so $\lambda v^T v > 0$, and since $v^T v > 0$, we have $\lambda > 0$. So we proved that all the eigenvalues of A are positive.

(a) $\Rightarrow$ (c) We now assume that all the eigenvalues of A are positive. Since A is symmetric, there exists an orthogonal n-by-n matrix V (such that $V^T V = I$) and a diagonal matrix D such that $A = VDV^T$ where D has the eigenvalues of A on its diagonal. Since eigenvalues of A are positive, we consider $D^{\frac{1}{2}}$ which has the square roots of the eigenvalues of A on its diagonal. Then, if we call $Q = (VD^{\frac{1}{2}}V^T)$, we have

$$A = VDV^T = VD^{\frac{1}{2}}D^{\frac{1}{2}}V^T = (VD^{\frac{1}{2}}V^T)(VD^{\frac{1}{2}}V^T) = QQ^T.$$

(Note that Q is invertible, since V and $D^{\frac{1}{2}}$ are.) We proved that there exists an invertible n-by-n matrix Q such that $A = QQ^T$.

Remarks Given a symmetric positive definite matrix A, there are plenty of matrices Q that satisfy $A = QQ^T$. In our proof above, we have chosen to use the (unique) (symmetric positive definite) square root of A: if A is symmetric positive definite, then we know that there exists a unique symmetric positive definite Q such that $A = Q^2$. (This is equivalent to $A = QQ^T$ since Q is symmetric.) Another choice for Q would have been to take the Cholesky factor of A: if A is symmetric positive definite, then there exists a (unique) lower triangular matrix Q (with positive diagonal elements) such that $A = QQ^T$. There were plenty of other valid choices for Q.

Problem 3

(a) Let A be an m-by-n matrix. Prove that if the matrix A^TA is invertible, then the matrix

$$I - A(A^TA)^{-1}A^T$$

is symmetric positive semidefinite. [10 points]

(b) In addition, let B be an m-by-p matrix. Prove that if A^TA and B^TB are invertible and if the ranges of A and B do not share a nontrivial subspace, then the matrix

$$B^T(I - A(A^TA)^{-1}A^T)B$$

is invertible. [10 points]

Solution 1 (based on the orthogonal projection in (a))

(a) We should be able to recognize the following facts: (1) Since A^TA is invertible, then A has full column rank and $m \geq n$. (2) $I - A(A^TA)^{-1}A^T$ is nothing else than the orthogonal projection on the orthogonal complement of Span(A), an orthogonal projection is a symmetric operator and its eigenvalues are 0 or 1. So the claim is indeed correct: $I - A(A^TA)^{-1}A^T$ is symmetric positive semidefinite. The question asks to prove this fact.

We call $P = I - A(A^TA)^{-1}A^T$.

First, we check that P is symmetric:

$$P^T = (I - A(A^TA)^{-1}A^T)^T = I - A(A^TA)^{-T}A^T = I - A(A^TA)^{-1}A^T = P.$$

Here we used the fact that $(A^TA)^{-1}$ is symmetric.

Second, we check that $P^2 = P$:

$$\begin{aligned} P^2 &= (I - A(A^TA)^{-1}A^T)^2 \\ &= I - 2A(A^TA)^{-1}A^T + A(A^TA)^{-1}A^TA(A^TA)^{-1}A^T \\ &= I - A(A^TA)^{-1}A^T = P. \end{aligned}$$

Since $P^2 = P$, we have $P(P - I) = 0$, so the eigenvalues of P are either 0 or 1.

P is symmetric with eigenvalue either 0 or 1 so P is symmetric positive semidefinite.

(b) Some remarks about this problem. Since A^TA is invertible, then $n \leq m$ and A is full column rank. Since B^TB is invertible, then $p \leq m$ and B is full column rank. We also know that the ranges of A and B do not have common nontrivial subspaces so, in other words, $\text{Range}(A) \cap \text{Range}(B) = \{0\}$. So we see that we must have $n + p \leq m$. Also, as explained above, $I - A(A^TA)^{-1}A^T$ is the orthogonal projection onto the orthogonal complement of Range(A). And we call it P.

We are to prove that B^TPB is invertible. We note, since $P^2 = P$ and P is symmetric, that $B^TPB = (PB)^T(PB)$. We therefore are to prove that $(PB)^T(PB)$ is invertible, or equivalently, that PB is full rank. We know that $\text{Null}(P) = \text{Range}(A)$. So, since $\text{Range}(A) \cap \text{Range}(B) = \{0\}$, we get that $\text{Null}(P) \cap \text{Range}(B) = \{0\}$, and so the matrix PB is full rank.

Solution 2 (using singular value decomposition for (a) with projection for (b))

(a) As before, we observe that A has full column rank with $m \geq n$. Hence, using a singular value decomposition, we can write $A = U\Sigma V^T$ so that $A^T A = V\Sigma^T\Sigma V^T$ and

$$\begin{aligned} A(A^T A)^{-1}A^T &= U\Sigma V^T V \left(\Sigma^T\Sigma\right)^{-1} V^T V\Sigma^T U^T \\ &= U\Sigma \left(\Sigma^T\Sigma\right)^{-1} \Sigma^T U^T = ULU^T \end{aligned}$$

where $L = \Sigma \left(\Sigma^T\Sigma\right)^{-1} \Sigma^T$ has a unit block n-by-n. Hence, it follows that

$$I - ULU^T = U(I - L)U^T$$

is positive semidefinite.

(b) The range of B is disjoint from the range of A, so that no nonzero element of the range of B is orthogonal to the nullspace of A^T. Combined with the fact that B has the trivial nullspace by $B^T B$ being invertible and that $I - A(A^T A)^{-1}A^T$ is the orthogonal projection operator onto the nullspace of A^T, it follows that $(I - A(A^T A)^{-1}A^T)B$ has the trivial nullspace, and thus

$$B^T(I - A(A^T A)^{-1}A^T)B = ((I - A(A^T A)^{-1}A^T)B)^T(I - A(A^T A)^{-1}A^T)B$$

has the trivial nullspace as well and is therefore invertible.

Problem 4

A square matrix N is called nilpotent if $N^m = 0$ for some positive integer m.

(a) Is the sum of two nilpotent matrices nilpotent? [5 pts]
If yes, prove it. If not, give a counterexample.

(b) Is the product of two nilpotent matrices nilpotent? [5 pts]
If yes, prove it. If not, give a counterexample.

(c) Prove that all eigenvalues of a nilpotent matrix are zero. [5 pts]

(d) Prove that the only nilpotent matrix that is diagonalizable is the zero matrix. [5 pts]

Solution

(a) No, the sum of two nilpotent matrices is not nilpotent, in general: the two matrices:

$$N_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } N_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

are nilpotent (with $m = 2$), but their sum:

$$S = N_1 + N_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is not nilpotent ($S^{2m-1} = S$ and $S^{2m} = I$ for all integers $m \geq 1$).

(b) No, the product of two nilpotent matrices is not nilpotent, in general: the two matrices N_1 and N_2 from part (a) are nilpotent, but both of their products:

$$P_{12} = N_1 N_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } P_{21} = N_2 N_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

are not nilpotent ($P^m = P$ for all integers $m \geq 1$).

(c) Let λ be an eigenvalue of N and x an associated nonzero eigenvector: $Nx = \lambda x$. Because N is nilpotent, we have $N^m = 0$ for some positive integer $m \geq 1$ and thus $N^m x = \lambda^m x = 0$ for $x \neq 0$. Hence, it follows that $\lambda^m = 0$ and thus $\lambda = 0$; because λ was any eigenvalue of N, *all* eigenvalues of N must be zero.
Alternate way. We know that, for any polynmial p and any square matrix A, if $p(A) = 0$ and if λ is an eigenvalue of A, then we have $p(\lambda) = 0$. Since N is nilpotent, we have $N^m = 0$ for some positive integer $m \geq 1$. Let λ be an eigenvalue of A. Then $\lambda^m = 0$, so $\lambda = 0$; because λ was any eigenvalue of N, *all* eigenvalues of N must be zero.

(d) Let N be nilpotent and diagonalizable. Because N is diagonalizable, it follows that there exists an invertible matrix V and a diagonal matrix D such that $N = VDV^{-1}$, where D contains the eigenvalues of N on its diagonal. Hence, since N is nilpotent and because we proved in (c) that all the eigenvalues of N are zero, the diagonal matrix D must be zero and thus $N = VDV^{-1}$ must be zero as well.

Problem 5

In this problem, you are asked to prove that two real symmetric matrices commute if and only if they are diagonalizable in a common orthonormal basis. We suggest the following path.

Let A and B be two real symmetric matrices and show each of the following. [5 points each]

(a) If A and B are diagonalizable in a common orthonormal basis, then A and B commute.

(b) If A and B commute, and if λ is an eigenvalue of A, then the eigenspace E_λ of A that is associated with the eigenvalue λ is invariant under B.

(c) If A and B commute, then A and B have at least one common eigenvector.

(d) If A and B commute, then A and B are diagonalizable in a common orthonormal basis.

Solution

(a) Let A and B be two n-by-n real symmetric matrices that are diagonalizable in a common orthonormal basis. Let V be the matrix of the vectors of this orthonormal basis. Then V is an orthogonal n-by-n matrix V (such that $V^TV = I$) and there exists a diagonal matrix D_A and a diagonal real matrix D_B such that $A = VD_AV^T$ and $B = VD_BV^T$. Now, we have

$$AB = VD_AV^TVD_BV^T = VD_AD_BV^T = VD_BD_AV^T = VD_BV^TVD_AV^T = BA,$$

where we have used the fact that $V^TV = I$ and the fact that two diagonal matrices always commute (so that $D_AD_B = D_BD_A$).

(b) Now we assume that A and B commute. Let λ be an eigenvalue of A, and E_λ, the eigenspace of A associated with the eigenvalue λ. Let x in E_λ, then $Ax = \lambda x$, so $BAx = \lambda Bx$, so, (since A and B commute,) $ABx = \lambda Bx$, so we see that Bx is in E_λ. So E_λ is invariant under B.

(c) We continue to assume that A and B commute. Let λ be an eigenvalue of A, and E_λ, the eigenspace of A associated with the eigenvalue λ. Since E_λ is invariant under B (see part(b)) and B is real symmetric, the restriction of B to E_λ is a real symmetric linear operator from E_λ to E_λ and so it has a real eigenvalue, say μ, with eigenvector v. (v, μ) is an eigencouple of the restriction of B to E_λ, but it is also an eigencouple of B. So, in fine, we have (v, μ) eigencouple of B, with v in E_λ, so v is also an eigenvector of A, associated with eigenvalue λ. We have found v a common eigenvector for A and B.

(d) Let A and B be two n-by-n real symmetric matrices that commute.

By (c), we know that there exists a common eigenvector for A and B. We can take this eigenvector of unit norm and we call it v_1. It is associated with the eigenvalue λ_1 for A and the eigenvalue μ_1 for B.

Now we consider $V_1 = \text{Span}(v_1)^\perp$ and we note that V_1 is invariant under A and under B.

(Proof of this claim for A (ditto for B): let x be in $V_1 = \text{Span}(v_1)^\perp$ then $v_1^T(Ax) = \lambda_1 v_1^T x$, but $v_1^T x = 0$ since x is in $v_1^\perp$, so $v_1^T(Ax) = 0$ so Ax is in $V_1 = \text{Span}(v_1)^\perp$. So V_1 is invariant under A.)

Since V_1 is invariant under A and A is real symmetric, A_1, the restriction of A to V_1, is a real symmetric linear operator from V_1 to V_1. Since V_1 is invariant under B and B is real symmetric, B_1, the restriction of B to V_1, is a real symmetric linear operator from V_1 to V_1. Also note that, since A and B commute, their restrictions A_1 and B_1 commute.

Now, we can use (c) on A_1 and B_1 to prove that A_1 and B_1 share a common eigenvector. We take this eigenvector of unit norm, and call it v_2. v_2 is a common eigenvector for A and B. Note that v_2 is orthogonal to v_1 since v_2 is in $V_1 = v_1^\perp$.

Then we consider $V_2 = \text{Span}(v_1, v_2)^\perp$, this is an invariant subspace for A and for B, so we consider the restrictions of A and B to V_2, and so on.

This process ends when we have constructed $(v_1, v_2, \ldots, v_n)$ n common eigenvectors of A and B which also form an orthonormal basis. We therefore have diagonalized A and B in a common orthonormal basis.

Problem 6

Let V be a 4-dimensional vector space over $\mathbb{R}$, let $\mathcal{L}(V,V)$ be the set of all linear mappings from V to V, and let $T\colon V \to V$ be a linear operator with minimal polynomial $\mu_T(x) = x^2+1$. Determine, with a proof, the dimension of the following subspace:

$$U(T) := \{S \in \mathcal{L}(V,V) \mid ST = TS\}. \qquad [20 \text{ points}]$$

Solution We begin with a few general comments.

- Firstly, we note that U is a subspace. The proof is trivial but it is good to observe this to ensure that the question for the dimension of this subspace makes sense.
- Secondly, we know that the homothetic linear transformations, which are simply multiplications by a scalar λ or of the form λI, commute with any operator so that, for all scalars λ, $\lambda I \in U$ and the dimension of U is at least 1.
- Thirdly, clearly T is in U since T commutes with itself. So $\operatorname{Span}(I,T)$ is in U. Also T is not of the form λI (otherwise, from $T^2 + I = 0$, we would get $\lambda^2 + 1 = 0$ which is not possible), so the list (I,T) is linearly independent, and so the dimension of U is at least 2.
- Fourthly, we can also observe that any polynomial of T, $p(T)$, commutes with T. However since $T^2 + I = 0$, for all p, $p(T)$ is in $\operatorname{Span}(I,T)$, so we do not find other dimensions (than the two we already have) with this observation.
- Fifthly, since U is in $\mathcal{L}(V,V)$, the dimension of U is at most 16.

Next, since the minimal polynomial of T is of the form $\mu(x) = x^2 + 1$, there exists a basis such that T is of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Henceforward, we will work in this basis. Let us use brute force. On the one hand:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = \begin{pmatrix} e & f & g & h \\ -a & -b & -c & -d \\ m & n & o & p \\ -i & -j & -k & -l \end{pmatrix}.$$

On the other hand:

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a & -d & c \\ -f & e & -h & g \\ -j & i & -l & k \\ -n & m & -p & o \end{pmatrix}.$$

Writing $TS = ST$, we get the following 16 equalities:

$$\begin{pmatrix} e = -b & a = f & d = -g & c = h \\ f = a & e = -b & h = c & g = -d \\ m = -j & i = n & l = -o & k = p \\ i = n & m = -j & p = k & l = -o \end{pmatrix}$$

Hence, we see that for S to commute with T, S has to have the following matrix representation (in the basis we already have chosen):

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ i & j & k & l \\ -j & i & -l & k \end{pmatrix}.$$

It follows that a basis for U could be:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right\}.$$

Hence, the dimension of U is 8. In particular, we see that the matrices of the form λI are in U, which is a good thing since we knew they had to be there.

Alternate Way We forget about the real vector space and consider a complex vector space first.
The eigenvalues of T are $\pm i$ with geometric multiplicities 2. We consider E_i, the eigenspace associated with i, and E_{-i}, the eigenspace associated with $-i$. We have $V = E_i \oplus E_{-i}$. We consider (v_1, v_2, v_3, v_4) a basis of V such that v_1 and v_2 are eigenvectors of T of eigenvalue i, and v_3 and v_4 are eigenvectors of T of eigenvalue $-i$.
We note that, if S commutes with T, then E_i is invariant under S and E_{-i} is invariant under S. Proof: Let v in E_i, then $Tv = iv$, so $STv = iSv$, so, $TSv = iSv$, so Sv in E_i.
Reciprocally, if S is such that E_i is invariant under S and E_{-i} is invariant under S, then S commutes with T. Proof: Let x in V. There exists x_1, x_2, x_3, and x_4 in $\mathbb{C}$ such that $x = x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4$. On the one hand, $STx = iS(x_1v_1 + x_2v_2) - iS(x_3v_3 + x_4v_4)$. On the other hand $TSx = TS(x_1v_1 + x_2v_2) + TS(x_3v_3 + x_4v_4)$. Since $x_1v_1 + x_2v_2$ in E_i, and E_i is invariant by S, we know that $S(x_1v_1 + x_2v_2)$ is in E_i and so $TS(x_1v_1 + x_2v_2) = iS(x_1v_1 + x_2v_2)$. Similarly for $-i$. All in all, we find that $TSx = iS(x_1v_1 + x_2v_2) - iS(x_3v_3 + x_4v_4)$. And so $TS = ST$.
We understand that S commutes with T if and only if E_i is invariant under S and E_{-i} is invariant under S.
The dimension of E_i is 2, so the dimension of $\mathcal{L}(E_i, E_i)$ is 4. The dimension of E_{-i} is 2, so the dimension of $\mathcal{L}(E_{-i}, E_{-i})$ is 4. The dimension of the subspace of the operators that leaves E_i and E_{-i} invariant is therefore 8.
The reasonning above was made in a complex vector space. We find that the subspace of the operators that commutes with T was a subspace of dimension 8. All real operators in this subspace also commutes with T and they form a subspace in the real vector space of dimension 8.