University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 23, 2015

Name: ____

Exam Rules:

- This exam lasts 4 hours and consists of 6 problems worth 20 points each.
- Each problem will be graded, and your final score will count out of 120 points.
- You are not allowed to use your books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen "essay-style" using full sentences and correct mathematical notation.
- Justify all your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce an independent proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

1.	____________	4.	____________
2.	____________	5.	____________
3.	____________	6.	____________
		Total	____________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

Find a basis for the intersection of the subspace of $\mathbb{R}^4$ spanned by $(1, 1, 0, 0)$, $(0, 1, 1, 0)$, $(0, 0, 1, 1)$ and the subspace spanned by $(1, 0, t, 0)$, $(0, 1, 0, t)$, where t is given. [20 points]

Problem 2

Let A be a real $m \times n$ matrix and b be a real m-dimensional vector. Prove the following.

(a) If the equation $Ax = b$ is consistent, then there exists a unique vector $p \in \text{Row}\, A$ such that $Ap = b$. [10 points]

(b) The equation $Ax = b$ is consistent if and only if the vector b is orthogonal to every solution of $A^T y = 0$. [10 points]

Problem 3

Let A and B be $n \times n$ matrices. Prove or disprove each of the following. [5 points each]

(a) If A and B are diagonalizable, then so is $A + B$.

(b) If A and B are diagonalizable, then so is AB.

(c) If $A^2 = A$, then A is diagonalizable.

(d) If A^2 is diagonalizable, then so is A.

Problem 4

Let A, B, and C represent three real $n \times n$ matrices, where A and B be symmetric positive definite (spd) and C be invertible. Prove that each of the following is spd. [5 points each]

(a) A^{-1}

(b) $A + B$

(c) $C^T AC$

(d) $A^{-1} - (A + B)^{-1}$

Problem 5

Let $\mathcal{P}_2[0,2]$ represent the set of polynomials with real coefficients and of degree less than or equal to 2, defined on $[0,2]$. For $p = (p(t)) \in \mathcal{P}_2$ and $q = (q(t)) \in \mathcal{P}_2$, define

$$\langle p, q\rangle := p(0)q(0) + p(1)q(1) + p(2)q(2).$$

(a) Verify that $\langle p, q\rangle$ is an inner product. [4 points]

(b) Let T represent the linear transformation that maps an element $p \in \mathcal{P}_2$ to the closest element of the span of the polynomials 1 and t in the sense of the norm associated with the inner product. Find the matrix A of T in the standard basis of $\mathcal{P}_2$.
(Note: the standard basis of $\mathcal{P}_2$ is $\{1, t, t^2\}$.) [10 points]

(c) Is A symmetric? Is T self-adjoint? Do these facts contradict each other? [3 points]

(d) Find the minimal polynomial of T. [3 points]

Problem 6

Suppose that A is an $m \times n$ matrix and B is an $n \times m$ matrix, and write I_m for the $m \times m$ identity matrix. Show that if $I_m - AB$ is invertible, then so is $I_n - BA$. [20 points]