University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 13, 2014

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________		

Total ________

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Applied Linear Algebra Preliminary Exam Committee:
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1. Assume the following general definition for a real positive semidefinite matrix: an $n \times n$ real matrix A is said to be *positive semidefinite* if and only if, for all vector x in $\mathbb{R}^n$, $x^T Ax \geq 0$. In particular, this definition allows real matrices which are *not symmetric* to be *positive semidefinite*.

   (a) Prove that if A and B are real symmetric positive semidefinite matrices and matrix A is nonsingular, then AB has *only* real nonnegative eigenvalues. (10 pts)

   (b) Provide a counterexample showing that the requirement that the matrices are symmetric cannot be dropped. (10 pts)

Solution

(a) Since A is symmetric positive definite, $A^{1/2}$ and $A^{-1/2}$ are well defined. The matrix AB has the same eigenvalues as the matrix $A^{-1/2}ABA^{1/2} = A^{1/2}BA^{1/2}$. The latter matrix is selfadjoint and positive semidefinite, so it has real nonnegative eigenvalues.

Note: The result also holds if we remove the assumption of A to be nonsingular. In other words, A and B only need to be two n–by–n symmetric positive semidefinite matrices. The proof gets a little trickier though.

(b) One needs to provide positive semidefinite matrices A and B, A nonsingular, such that AB has an eigenvalue which is not "real and nonnegative". Given question (a) we understand that either A or B (or both) have to be non-symmetric. To create a positive semidefinite matrix A, one simply takes a symmetric positive definite matrix H and then add an antisymmetric matrix S, then $A = H + S$ is positive semidefinite matrix.

In our case, we can take $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

In this case A is positive semidefinite nonsingular, B is positive semidefinite, and AB does not have real nonnegative eigenvalues.

2. (a) Suppose A and B are real-valued symmetric $n \times n$ matrices. Show that $|\text{trace}\ (AB)| \leq \sqrt{\text{trace}\ (A^2)}\sqrt{\text{trace}\ (B^2)}$. What are the conditions for equality to hold? (10 pts)

   (b) Let A be a real $m \times n$ matrix. Show that

$$\sqrt{\text{trace}(AA^T)} \leq \text{trace}\left(\sqrt{AA^T}\right).$$

   When does equality hold ? (10 pts)

Solution

(a) By the Cauchy-Schwarz Theorem,

$$|\text{trace}\ (AB)| = \left|\sum_{i,j} a_{ij}b_{ij}\right| \leq \sqrt{\sum_{i,j} a_{ij}^2}\sqrt{\sum_{i,j} b_{ij}^2} = \sqrt{\text{trace}\ (A^2)}\sqrt{\text{trace}\ (B^2)}.$$

For equality to hold, one of the matrices has to be a scalar multiple of the other.

(b) Let $AA^T = P^TDP$, where D represents a nonnegative diagonal matrix and P represents an orthogonal matrix. Then

$$\text{trace}(AA^T) = \text{trace}(D) = \sum_i \lambda_i \leq (\sum_i \sqrt{\lambda_i})^2 = (\text{trace}(D^{1/2}))^2 = (\text{trace}((AA^T)^{1/2}))^2.$$

The fact that $\sum_i \lambda_i \leq (\sum_i \sqrt{\lambda_i})^2$ comes from developing the square on the right side. Equality holds if and only if D has at most one nonzero entry, so AA^T has at most one nonzero eigenvalue, so A has at most one nonzero singular value.

3. Let

$$\begin{array}{rcl} f : \mathcal{M}_n(\mathbb{R}) & \longrightarrow & \mathcal{M}_n(\mathbb{R}) \\ A & \longmapsto & A^T \end{array}$$

(a) What are the eigenvalues of f? (10 pts)

(b) Is f diagonalizable? If yes, give a basis of eigenvectors. If no, give as many linearly independent eigenvectors as possible. (10 pts)

Solution

It is clear that $f^2 = I$, therefore $p(x) = (x-1)(x+1)$ is such that $p(f) = 0$. This implies that the eigenvalues of f are part of the set $\{1, -1\}$. Also $p(f) = 0$ implies that f is diagonalizable since p only has single roots.

Now it is clear that any symmetric matric is eigenvector associated with eigenvalue 1, and that an eigenvector associated with eigenvalue 1 is a symmetric matrix. If we call the subspace of symmetric matrices, $\mathcal{S}_n$, and E_1 the eigenspace of f associated with eigenvalue 1, we have $\mathcal{S}_n = E_1$.

It is also clear that any antisymmetric matric is eigenvector associated with eigenvalue -1, and that an eigenvector associated with eigenvalue -1 is an antisymmetric matrix. If we call the subspace of antisymmetric matrices, $\mathcal{A}_n$, and E_{-1} the eigenspace of f associated with eigenvalue -1, we have $\mathcal{A}_n = E_{-1}$.

We know that

$$\mathcal{M}_n = \mathcal{S}_n \oplus \mathcal{A}_n.$$

Therefore we can diagonalize f by taking a basis of $\mathcal{S}_n$ and a basis of $\mathcal{A}_n$ to form a basis of $\mathcal{M}_n$.

4. Define the $n \times n$ matrix

$$A_n = \begin{bmatrix} a+b & b & b & \dots & b & b \\ a & a+b & b & \ddots & b & b \\ a & a & a+b & \ddots & b & b \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ a & a & a & \ddots & a+b & b \\ a & a & a & \dots & a & a+b \end{bmatrix}$$

(a) Compute $D_n = \det(A_n)$. (10 pts)

(b) Give the value of D_n for $n = 10$, $a = 2$, and $b = -1$. (10 pts)

Solution

We perform (in this order) $L_n \leftarrow L_n - L_{n-1}$, then $L_{n-1} \leftarrow L_{n-1} - L_{n-2}$, ... and finally $L_2 \leftarrow L_2 - L_1$. (These transformations do not change the value of the determinant.) We get

$$D_n = \begin{vmatrix} a+b & b & b & \dots & b & b \\ -b & a & 0 & \ddots & 0 & 0 \\ 0 & -b & a & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & a & 0 \\ 0 & 0 & 0 & \dots & -b & a \end{vmatrix}.$$

We develop with respect to last column and get

$$D_n = (-1)^{n-1} b \begin{vmatrix} -b & a & 0 & \ddots & 0 \\ 0 & -b & a & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & a \\ 0 & 0 & 0 & \dots & -b \end{vmatrix} + a \begin{vmatrix} a+b & b & b & \dots & b \\ -b & a & 0 & \ddots & 0 \\ 0 & -b & a & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & -b & a \end{vmatrix}.$$

And so, we get

$$D_n = b^n + aD_{n-1}.$$

We have

$$D_1 = a + b.$$

(Note: We could get D_1 from $D_1 = b + aD_0$ if we define D_0 to be 1.)

So we get

$$D_2 = b^2 + aD_1 = b^2 + ab + a^2.$$

Quick check:

$$D_2 = \begin{vmatrix} a+b & b \\ a & a+b \end{vmatrix} = (a+b)^2 - ab = b^2 + ab + a^2.$$

So we get

$$D_3 = b^3 + aD_2 = b^3 + ab^2 + a^2b + a^3$$

Pursuing in an identical manner, we get

$$D_n = b^n + ab^{n-1} + \ldots + a^{n-1}b + a^n = \sum_{k=0}^{n} a^k b^{n-k}.$$

We can simplify by noticing that

$$(a-b)(b^n + ab^{n-1} + \ldots + a^{n-1}b + a^n) = a^{n+1} - b^{n+1}.$$

So, if $a \neq b$, we have

$$D_n = \frac{a^{n+1} - b^{n+1}}{a-b}.$$

And, if $a = b$, we get

$$D_n = (n+1)a^n.$$

(And we check that the latter expression for $a = b$ is the limit of the expression for $a \neq b$ when b goes to a.)

For $n = 10$, $a = -1$, and $b = 2$, we get

$$\frac{(-1)^{11} - (2)^{11}}{(-1) - 2} = \frac{2049}{3} = 683.$$

5. Suppose that u and v are vectors in a real inner product space V.

(a) Prove that

$$(||u|| + ||v||)\frac{\langle u, v\rangle}{||u||\,||v||} \leq ||u + v||. \quad \text{(10 pts)}$$

(b) Prove or disprove the following identity:

$$(||u|| + ||v||)\frac{|\langle u, v\rangle|}{||u||\,||v||} \leq ||u + v||. \quad \text{(10 pts)}$$

Solution

(a) Case 1: $\langle u, v\rangle \leq 0$. The inequality follows trivially since a norm is nonnegative. Thus, the leftside is no more than 0 while the right side is no less than 0.

Case 2: $\langle u, v\rangle > 0$. Squaring the left side we have

$$\begin{aligned}
(||u|| + ||v||)^2\frac{\langle u, v\rangle\langle u, v\rangle}{||u||^2||v||^2} &\leq (||u||^2 + ||v||^2 + 2||u||\,||v||)\frac{\langle u, v\rangle||u||\,||v||}{||u||^2\,||v||^2} && (1)\\
&= \frac{||u||}{||v||}\langle u, v\rangle + \frac{||v||}{||u||}\langle u, v\rangle + 2\langle u, v\rangle && (2)\\
&= \frac{||u||}{||v||}||u||\,||v|| + \frac{||v||}{||u||}||u||\,||v|| + 2\langle u, v\rangle && (3)\\
&= ||u + v||^2. && (4)
\end{aligned}$$

Both (1) and (3) are obtained by applying the Cauchy-Schwarz inequality to $\langle u, v\rangle$, while (2) and (4) are obtained by simplifying.

(b) Let $u = (1, 0)$, $v = (-1, 0)$, and use a Euclidean inner product (dot product). Then the left side of the inequality becomes $(1 + 1)\frac{1}{(1)(1)} = 1$ while the right side is 0. (Note: one can also use one-dimensional vector: $u = (1)$, $v = (-1)$.)

6. Let V be a vector space. Let $f \in \mathcal{L}(V)$. Let p be a projection (so $p \in \mathcal{L}(V)$ and is such that $p^2 = p$). Prove that

$$\text{Null}(f \circ p) = \text{Null}(p) \oplus (\text{Null}(f) \cap \text{Range}(p)). \quad (20 \text{ pts})$$

Solution

Firstly, we would like to prove that

$$\text{Null}(p) \oplus (\text{Null}(f) \cap \text{Range}(p)) \subset \text{Null}(f \circ p).$$

Note: We recall that if A, B and C are subspaces, to prove that $A + B \subset C$, we just need to prove that $A \subset C$ and $B \subset C$.

$\boxed{\text{Null}(p) \subset \text{Null}(f \circ p)}$ Let $x \in \text{Null}(p)$, then $p(x) = 0$, so $(f \circ p)(x) = 0$, so $x \in \text{Null}(f \circ p)$.

$\boxed{\text{Null}(f) \cap \text{Range}(p) \subset \text{Null}(f \circ p)}$ Let $x \in \text{Null}(f) \cap \text{Range}(p)$. Since $x \in \text{Range}(p)$, there exists y such that $x = p(y)$. Since $x \in \text{Null}(f)$, we have $f(x) = 0$. Now let us look at $(f \circ p)(x)$. (Note: we want to prove that $(f \circ p)(x) = 0$.) We have $(f \circ p)(x) = (f \circ p)(p(y)) = f(p^2(y)) = f(p(y)) = f(x) = 0$, We have used the facts that $1 \to 2$: $x = p(y)$, $3 \to 4$: $p^2 = p$, $4 \to 5$: $p(y) = x$, $5 \to 6$: $f(x) = 0$. This proves that $x \in \text{Null}(f \circ p)$.

We proved that

$$(\text{Null}(p) + (\text{Null}(f) \cap \text{Range}(p))) \subset \text{Null}(f \circ p).$$

Secondly, we would like to prove that

$$\text{Null}(f \circ p) \subset \text{Null}(p) \oplus (\text{Null}(f) \cap \text{Range}(p)).$$

Let $x \in \text{Null}(f \circ p)$, we can write x as

$$x = (x - p(x)) + p(x),$$

where

(a) $\boxed{(x - p(x)) \in \text{Null}(p).}$ Indeed, $p(x - p(x)) = p(x) - p^2(x)$, but $p = p^2$ so $p(x - p(x)) = 0$, so $(x - p(x)) \in \text{Null}(p)$.

(b) $\boxed{p(x) \in \text{Null}(f) \cap \text{Range}(p).}$ It is a fact that $p(x) \in \text{Range}(p)$. Moreover, since $x \in \text{Null}(f \circ p)$, we have that $(f \circ p)(x) = 0$, which proves that $p(x) \in \text{Null}(f)$. So $p(x) \in \text{Null}(f) \cap \text{Range}(p)$.

Therefore we have that

$$\text{Null}(f \circ p) \subset \text{Null}(p) + (\text{Null}(f) \cap \text{Range}(p))\,.$$

At this point, we proved that

$$\text{Null}(f \circ p) = \text{Null}(p) + (\text{Null}(f) \cap \text{Range}(p))\,.$$

It remains to prove that the sum is direct. Let $x \in \text{Null}(p) \cap (\text{Null}(f) \cap \text{Range}(p))$, then $x \in \text{Range}(p)$, so there exits $u \in V$ such that $x = p(u)$, but $x \in \text{Null}(p)$, so $p(x) = 0$, so $p^2(u) = 0$, but $p^2 = p$, so $p(u) = 0$, so $x = 0$. We proved that $\text{Null}(p) \cap (\text{Null}(f) \cap \text{Range}(p)) = \{0\}$ so the sum in the previous paragraph is direct.

We are done and we can conclude that

$$\text{Null}(f \circ p) = \text{Null}(p) \oplus (\text{Null}(f) \cap \text{Range}(p))\,.$$

7. (a) Let $n \in \mathbb{N}\backslash\{0,1\}$ (so $n \geq 2$) and $A \in \mathcal{M}_n(\mathbb{C})$ such that $\text{rank}(A) = 1$. Prove that A is diagonalizable if and only if $\text{trace}(A) \neq 0$. (10 pts)

   (b) Let $a_1, \dots a_n \in \mathbb{C}\backslash\{0\}$, (so the a_i's are nonzero complex numbers,) and A such that $A = \left(\frac{a_i}{a_j}\right)_{1\leq i,j\leq n}$. (This means that the entry (i,j) of A is $\frac{a_i}{a_j}$.) Show that A is diagonalizable. Give a basis of eigenvectors (with the associated eigenvalues) for A. (10 pts)

Solution

(a) First we note that $\text{rank}(A) = 1 \Leftrightarrow \dim(\text{Null}(A)) = n - 1$ (by the rank theorem). So, if $\text{rank}(A) = 1$ and $n \geq 2$, then $\dim(\text{Null}(A)) \geq 1$ and so 0 is an eigenvalue of A. We call ν_0 the geometric multiplicity of the eigenvalue 0, and μ_0 the algebraic multiplicity of the eigenvalue 0. We call E_0 the eigenspace associated with the eigenvalue 0. Now, since $\dim(\text{Null}(A)) = n - 1$, we have that $\dim(E_0) = n - 1$, or in other words, the geometric multiplicity of the eigenvalue 0, ν_0, is $n - 1$. We know that, for a given eigenvalue, the algebraic multiplicity is always greater than or equal to the geometric multiplicity. For the eigenvalue 0, this reads: $\nu_0 \leq \mu_0$. For a rank–1 matrix, there are therefore only two cases: either $\nu_0 = \mu_0 = n - 1$, or $\nu_0 = n - 1$, $\mu_0 = n$.

case $\nu_0 = \mu_0 = n - 1$ In this case, since $\mu_0 = n-1$, there has to exist another eigenvalue λ different from zero. (Because the sum of the algebraic multiplicities of the eigenvalues has to sum to n.) For that eigenvalue λ, the geometric multiplicity, ν_λ, is at least 1, but can be no more than 1 (because $\nu_0 = n - 1$ and the sum of the algebraic multiplicities of two distinct eigenvalues has to be less than n). So $\nu_\lambda = 1$. So we have $\nu_\lambda = 1$ and $\nu_0 = n - 1$, so A is diagonalizable. We also note that, in this case, $\text{trace}(A) = \lambda$, (the trace is the sum of the eigenvalues counted with their multiplicities,) and so, in this case, $\text{trace}(A) \neq 0$.

case $\nu_0 = n - 1, \mu_0 = n$ In this case, since $\mu_0 = n$, A only has the eigenvalue 0. We also have that A is not diagonaliable and that $\text{trace}(A) = 0$.

Starting from a rank–1 matrix, we found two possibilities. Either $\nu_0 = \mu_0 = n-1$, in which case, A is diagonaliable and $\text{trace}(A) \neq 0$. Or $\nu_0 = n-1, \mu_0 = n$, in which case, A is not diagonaliable and $\text{trace}(A) = 0$.

This enables us to conclude that for a rank–1 matrix

$$A \text{ is diagonalizable} \Leftrightarrow \text{trace}(A) \neq 0.$$

(b) We observe that the matrix is of rank 1. Indeed

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} \begin{pmatrix} \frac{1}{a_1} & \frac{1}{a_2} & \cdots & \frac{1}{a_{n-1}} & \frac{1}{a_n} \end{pmatrix}.$$

We also have trace$(A) = n$. So by the previous question, we see that A is diagonalizable (since trace$(A) \neq 0$. We also see that A has eigenvalue 0 with geometric multiplicity $n-1$ and eigenvalue n with geometric multiplicity 1.

$\boxed{\text{eigenvalue } 0}$ To find $n-1$ linearly independent eigenvectors associated with eigenvalue 0, we want to find a basis for the null space of A, which is same as null space of

$$\begin{pmatrix} \frac{1}{a_1} & \frac{1}{a_2} & \cdots & \frac{1}{a_{n-1}} & \frac{1}{a_n} \end{pmatrix}.$$

We have (for example) that x_1 is a leading variable, and that x_2, x_3 ,..., x_n are free variables. This gives for a general solution:

$$\begin{pmatrix} -\frac{a_1}{a_2}x_2 - \frac{a_1}{a_3}x_3 - \ldots - \frac{a_1}{a_{n-1}}x_{n-1} - \frac{a_1}{a_n x_n} \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = x_2 \begin{pmatrix} -\frac{a_1}{a_2} \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -\frac{a_1}{a_3} \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} + \ldots + x_{n-1} \begin{pmatrix} -\frac{a_1}{a_{n-1}} \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + x_n \begin{pmatrix} -\frac{a_1}{a_n} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

So a basis for E_0 is for example

$$v_1 = \begin{pmatrix} -a_1 \\ a_2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -a_1 \\ 0 \\ a_3 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \ldots \quad v_{n-2} = \begin{pmatrix} -a_1 \\ 0 \\ 0 \\ \vdots \\ a_{n-1} \\ 0 \end{pmatrix}, \quad v_{n-1} = \begin{pmatrix} -a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ a_n \end{pmatrix}.$$

$\boxed{\text{eigenvalue } n}$ We see that an eigenvector for eigenvalue n is for example

$$v_n = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}.$$

<u>Answer:</u> The above given $(v_1, \ldots, v_n)$ is a basis of $\mathbb{C}^n$ made of eigenvectors of A.