University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 13, 2014

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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1. Let A be a full column rank n–by–k matrix (so $k \leq n$) and b to be a column vector of size n. We want to minimize the squared Euclidean norm $L(x) := ||Ax - b||_2^2$ with respect to x.

   (a) Prove that, if $\text{rank}(A) = k$, then $A^T A$ is invertible.

   (b) Compute the gradient of $L(x)$.

   (c) Directly derive the normal equations by minimizing $L(x)$, and then provide the closed-form expression for x that minimizes $L(x)$.

   (d) We consider a QR factorization of A where Q is n-by-k and R is k-by-k. Show that an equivalent solution for x is $x = R^{-1}Q^T b$.

2. Let V be a real vector space.

   (a) Give the definition of a real inner product $\langle \cdot, \cdot \rangle$ over the vector space V. (That is the set of properties from the definition of of a real inner product.)

   We define $\|x\|$ as $\|x\| = \sqrt{\langle x, x \rangle}$.

   (b) From these two definitions, state and prove the Cauchy-Schwarz inequality.

   (c) Now, state and prove the triangular inequality.

   (d) Now, prove that $\|x\|$ is a norm.

3. Suppose A is a positive definite symmetric real n–by–n matrix and B is a real m–by–n matrix such that BB^T is positive definite. Prove that the matrix $B^T(BA^{-1}B^T)^{-1}B$ is symmetric positive definite.

4. Suppose A is a positive definite symmetric square real matrix and B is a symmetric square real matrix. Show that there exists a square real matrix C such that $C^T AC$ is the identity matrix and $C^T BC$ is a diagonal matrix.

5. Let $\mathcal{P}_n$ represent the real vector space of polynomials in x of degree less than or equal to n defined on $[0,1]$. Given a real number a, we define $Q_n(a)$ the subset of $\mathcal{P}_n$ of polynomials that have the real number a as a root.

   (a) Let a be a real number. Show that $Q_n(a)$ is a subspace of $\mathcal{P}_n$. Determine the dimension of that subspace and exhibit a basis.

   (b) Let the inner product in $\mathcal{P}_n$ be defined by $\langle p, q\rangle = \int_0^1 p(x)q(x)dx$. Determine the orthogonal complement of the subspace $Q_2(1)$ of $\mathcal{P}_2$.

6. Let $\mathbb{F}$ be a commutative field, let $(V, +, .)$ be a vector space over $\mathbb{F}$, let A and B be two subspaces of V, let A' be a subspace such that $A' \oplus (A \cap B) = A$ and let B' be a subspace such that $B' \oplus (A \cap B) = B$. Show that $A + B = (A \cap B) \oplus A' \oplus B'$.

7. Let $\mathbb{F}$ be a commutative field, let $(V,+,.)$ be a vector space over $\mathbb{F}$, let n be a natural number, let $(e_1,\ldots,e_n)$ be a linear independent list in V, let $\lambda_1,\ldots,\lambda_n$ be n scalars in $\mathbb{F}$, let $u=\sum_{i=1}^{n}\lambda_i e_i$, and let, for all $i=1,\ldots,n$, $v_i=u+e_i$. Show that $(v_1,\ldots v_n)$ is linearly dependent if and only $\sum_{i=1}^{n}\lambda_i=-1$.

8. What is the rank of

$$\begin{pmatrix} 1 & a & 1 & b \\ a & 1 & b & 1 \\ 1 & b & 1 & a \\ b & 1 & a & 1 \end{pmatrix}?$$

The rank is a function of a and b. You need to give the values of the rank for all values of $(a, b) \in \mathbb{R}^2$.