University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 14, 2013

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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1. Find the least squares solution of $Ax = b$ where

$$A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix}.$$

Solution

The linear least squares solution x is given by $x = (A^T A)^{-1} A^T b$.

$$A^T b = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ 6 & 42 \end{pmatrix} = 6 \begin{pmatrix} 1 & 1 \\ 1 & 7 \end{pmatrix}$$

$$(A^T A)^{-1} = \frac{1}{36} \begin{pmatrix} 7 & -1 \\ -1 & 1 \end{pmatrix}$$

$$x = (A^T A)^{-1}(A^T b) = \frac{1}{36} \begin{pmatrix} 7 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ -6 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 7 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -1/3 \end{pmatrix}.$$

2. Let $\mathbb{F}$ be a field. Let $\mathcal{P}_1$ denote the standard vector space of polynomials $f(t)$ with coefficients in the field $\mathbb{F}$ and having degree at most 1. Let $\mathcal{S} = \{1, t\}$ be the standard ordered basis of $\mathcal{P}_1$.

   (a) Define $T \in \mathcal{L}(\mathcal{P}_1)$ by

   $$T : p(t) = a + bt \mapsto q(t) = 5a - 2b + (4a - b)t.$$

   Construct the matrix $A = [T]_\mathcal{S}$ that represents T with respect to the basis $\mathcal{S}$. Is there an ordered basis $\mathcal{B}$ for $\mathcal{P}_1$ such that $[T]_\mathcal{B}$ is diagonal? If so, give such a basis and the corresponding matrix representation. If not, explain why not.

   (b) Replace T of part (a) by $S \in \mathcal{L}(\mathcal{P}_1)$ defined by

   $$S : p(t) = a + bt \mapsto q(t) = -a + b - bt,$$

   and repeat question (a).

Solution

(a) Since $T(1) = 5 + 4t$, the first column of $[T]_\mathcal{S}$ is $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$. Similarly, $T(t) = -2 - t$ implies the second column is $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$. So $A = [T]_\mathcal{S} = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix}$. A has eigenvalues 3 and 1 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, respectively. Since T has $2 = \dim(\mathcal{P}_1)$ distinct eigenvalues, T is diagonalizable with diagonalization

$$S^{-1}AS = D, \text{with } S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \text{and } D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the desired basis is $\mathcal{B} = \{1 + t, 1 + 2t\}$, for which $[T]_\mathcal{B} = D$.

(b) $A = [T]_\mathcal{S} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. This matrix is in Jordan form and is an elementary Jordan block that is not diagonal. Hence A is not diagonalizable. Therefore, there is no basis for which the corresponding matrix representation is diagonal.

3. Let A be a real matrix. A *generalized inverse* of a matrix A is any matrix G such that $AGA = A$. Prove each of the following:

   (a) If A is invertible, the unique generalized inverse of A is A^{-1}.

   (b) If G is a generalized inverse of $(X^T X)$, then

$$XGX^T X = X \ .$$

   (c) For any real symmetric matrix A, there exists a generalized inverse of A.

Solution

(a) $AA^{-1}A = IA = A$, so A^{-1} is a generalized inverse. If $AA^+A = A$, then $AA^+ = AA^+AA^{-1} = AA^{-1} = I$, so A^+ is the inverse of A.

(b) For arbitrary vector v, we can write $v = u + w$, where $u \in \operatorname{null} X^T$ and $w = X\lambda$. Then

$$v^T XGX^T X = (u^T + \lambda^T X^T)XGX^T X = \lambda^T X^T XGX^T X = \lambda^T X^T X = w^T X = v^T X.$$

Since v is arbitrary, $XGX^T X = X$.

(c) Since A is real symmetric, it is diagonalizable; so $A = P\Lambda P^T$, where P is orthogonal and Λ is diagonal real, with the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_n)$ on the diagonal. Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ where

$$\gamma_i = \begin{cases} \frac{1}{\lambda_i} & \text{if } \lambda_i \neq 0 \\ 0 & \text{if } \lambda_i = 0. \end{cases}$$

Let Γ be the diagonal matrix with γ along the diagonal. Let $G = P\Gamma P^T$. Since P is orthogonal, $P^T P = I$. Thus,

$$\begin{aligned} AGA &= P\Lambda P^T P\Gamma P^T P\Lambda P^T \\ &= P\Lambda\Gamma\Lambda P^T \\ &= P\Lambda P^T = A \end{aligned}$$

Thus G is a generalized inverse of A.

4. Let A be a real symmetric n-by-n matrix which is not just a scalar multiple of the identity matrix. Let $f(x) = (x-1)(x+6)^3$ and suppose that $f(A) = 0$ and the trace of A is 0.

   (a) Determine the minimal polynomial of A.

   (b) Determine the trace of A^2 as a function of n.

   (c) Show that n is a multiple of 7.

   (d) Determine the characteristic polynomial of A as a function of n.

Solution

Since A is real symmetric, its minimal polynomial has no repeated factors, and since $f(A) = 0$ the minimal polynomial divides $f(x)$. Since A is not a scalar times the identity, the minimal polynomial of A has to be exactly $p(x) = (x-1)(x+6) = x^2 + 5x - 6$.

Since $p(A) = 0$, we have that $A^2 = -5A + 6I$. So the trace of A^2 is $-5(\text{trace}(A)) + 6n = 6n$.

As eigenvalues of A, suppose 1 has multiplicity u and -6 has multiplicity v. (Since A is real symmetric, algebraic and geometric multiplicities are the same.)

On the one hand, we have $u + v = n$. (I.e., for any matrix, the sum of the algebraic multiplicities is always n or, since A is real symmetric, A is diagonalizable, and so the sum of the geometric multiplicities is n.) On the other hand, we know that $\text{trace}(A) = 0$ and we know that $\text{trace}(A)$ is the sum of the eigenvalues counting (algebraic – in the general case) multiplicities, therefore $u - 6v = 0$.

Solving $u + v = n$ and $u - 6v = 0$, a system of two linear equations in the two unknowns u and v, we find $u = \frac{6n}{7}$ and $v = \frac{n}{7}$, both of which are positive integers. So there is some positive integer k for which $n = 7k$, $u = 6k$, $v = k$. n is a multiple of 7.

The characteristic polynomial is

$$c_A(x) = (x-1)^{\frac{6}{7}n}(x+6)^{\frac{1}{7}n}.$$

$$c_A(x) = \left(x^7 - 21x^5 + 70x^4 - 105x^3 + 84x^2 - 35x + 6\right)^{\frac{n}{7}}.$$

5. Let U and W be subspaces of the finite-dimensional inner product space V.

(a) Prove that $U^\perp \cap W^\perp = (U+W)^\perp$.

(b) Prove that

$$\dim(W) - \dim(U \cap W) = \dim(U^\perp) - \dim(U^\perp \cap W^\perp).$$

Solution

Let $x \in U^\perp \cap W^\perp$. Then for any $u \in U$ and $w \in W$, $\langle x, u+w\rangle = \langle x, u\rangle + \langle x, w\rangle = 0$. Thus, $x \in (U+W)^\perp$, so $U^\perp \cap W^\perp \subset (U+W)^\perp$.

For any $y \in (U+W)^\perp$, and any $u \in U$ and $w \in W$, we have $u = u + 0 \in U + W$, so $\langle y, u\rangle = 0$. Similarly, $\langle y, w\rangle = 0$. Thus, $y \in U^\perp \cap W^\perp$. Thus, $(U+W)^\perp \subset U^\perp \cap W^\perp$. It follows that $(U+W)^\perp = U^\perp \cap W^\perp$, proving part (a).

Keep in mind that for finite-dimensional inner product spaces we know that $\dim(U^\perp) = \dim(V) - \dim(U)$. Then for the proof of (b) consider the following:

$$\begin{aligned}
\dim(U^\perp) \quad - &\quad \dim(U^\perp \cap W^\perp) = (\dim(V) - \dim(U)) - \dim\left((U+W)^\perp\right) \\
= &\quad \dim(V) - \dim(U) - (\dim(V) - \dim(U+W)) \\
= &\quad \dim(U) + \dim(W) - \dim(U \cap W) - \dim(U) \\
= &\quad \dim(W) - \dim(U \cap W), \quad \text{as desired.}
\end{aligned}$$

6. Let B be an n-by-n Hermitian matrix. Then B has real eigenvalues which we may order as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. For $\overline{0} \neq \mathbf{x} \in \mathbb{C}^n$, and using the usual 2-norm $\|\mathbf{x}\| = \|\mathbf{x}\|_2$, define the Rayleigh Quotient $\rho_B(\mathbf{x})$ for B by

$$\rho_B(\mathbf{x}) = \frac{\langle B\mathbf{x}, \mathbf{x}\rangle}{\langle \mathbf{x}, \mathbf{x}\rangle} = \frac{\mathbf{x}^* B\mathbf{x}}{\|\mathbf{x}\|^2}.$$

Prove the following:

(i) If B is an n-by-n Hermitian with eigenvalues as above, prove that $\lambda_1 = \max\{\rho_B(\mathbf{x}) : \mathbf{x} \in \mathbb{C}^n \text{ and } \|\mathbf{x}\| = 1\}$.

(ii) Let A be any $n \times n$ complex matrix with largest singular value σ_1. If $\|A\|_2 = \max\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{C}^n \text{ and } \|\mathbf{x}\| = 1\}$, show that

$$\|A\|_2 = \sigma_1.$$

Solution

First note that if $0 \neq k \in \mathbb{C}$ and $\overline{0} \neq \mathbf{x} \in \mathbb{C}^n$, then $\rho_B(k\mathbf{x}) = \rho_B(\mathbf{x})$. If we put $\mathcal{O} = \{\mathbf{x} \in \mathbb{C}^n : \|\mathbf{x}\| = 1\}$, then

$$\sup\{\rho_B(\mathbf{x}) : \overline{0} \neq \mathbf{x} \in \mathbb{C}^n\} = \sup\{\rho_B(\mathbf{x}) : \mathbf{x} \in \mathcal{O}\}.$$

Second, since B is hermitian, there is an orthonormal basis $\mathcal{B} = (v_1, \ldots, v_n)$ of eigenvectors so that $Bv_j = \lambda_j v_j$, for $j = 1, 2, \ldots, n$. If we put v_j in as the jth column of the $n \times n$ matrix P, then P is unitary ($P^* = P^{-1}$) and $P^*BP = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Since $\mathbf{y} \mapsto P\mathbf{y} = \mathbf{x}$ maps $\mathcal{O}$ to $\mathcal{O}$ in a one-to-one and onto manner, we have

$$\begin{aligned}
\sup\{\rho_B(\mathbf{x}) : \mathbf{x} \in \mathcal{O}\} &= \sup\{\mathbf{x}^* B\mathbf{x} : \mathbf{x} \in \mathcal{O}\} \\
= \sup\{(P\mathbf{y})^* B(P\mathbf{y}) : \mathbf{y} \in \mathcal{O}\} &= \sup\{\mathbf{y}^* \Lambda \mathbf{y} : \mathbf{y} \in \mathcal{O}\} \\
&= \sup\{\sum_{j=1}^{n} \lambda_j |y_j|^2 : (y_1, \ldots, y_n)^T \in \mathcal{O}\} \\
&\leq \sup\{\lambda_1 \sum_{j=1}^{n} |y_j|^2 : \sum_{j=1}^{n} |y_j|^2 = 1\} = \lambda_1.
\end{aligned}$$

So to prove part (i), we just need to find an $\mathbf{x} \in \mathcal{O}$ for which $\rho_B(\mathbf{x}) = \lambda_1$. Clearly $\mathbf{x} = v_1$ will work (with $\mathbf{y} = P^{-1}\mathbf{x} = (1, 0, \ldots, 0)^T$).

For part (ii), we note that $B = A^*A$ is hermitian, and we can adapt the notation of part (i) and use the fact that the largest eigenvalue of A^*A is $\lambda_1 = \sigma_1^2$ to obtain

$$\begin{aligned}
\|A\|_2 &= \max\{\|A\mathbf{x}\| : \mathbf{x} \in \mathbb{C}^n \text{ and } \|\mathbf{x}\| = 1\} \\
&= \max\{\sqrt{\mathbf{x}^* A^* A\mathbf{x}} : \mathbf{x} \in \mathcal{O}\} \\
&= \sqrt{\sigma_1^2} = \sigma_1. \text{(By part (i))}
\end{aligned}$$

7. Let T be a normal operator on a finite-dimensional complex inner product space V.

(a) Prove that T is self-adjoint if and only if its eigenvalues are all real.

(b) Prove that T is positive (i.e., positive semidefinite) if and only if all its eigenvalues are nonnegative.

Solution

Since T is normal, by the complex spectral theorem, there is an orthonormal basis $\{e_1, \dots, e_n\}$ of V consisting of eigenvectors of T, with corresponding eigenvalues $\lambda_1, \dots \lambda_n$. The matrix of T with respect to the basis $\{e_1, \dots, e_n\}$ is the diagonal matrix $D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$.

(a) T is self-adjoint if and only if $D = D^*$ if and only if $\lambda_j = \bar{\lambda}_j$ (i.e., λ_j is real) for all j.

(b) First suppose T is positive, so $\langle Tv, v\rangle \geq 0$ for all $v \in V$. Then, for each eigenpair (λ_j, e_j), $\langle Te_j, e_j\rangle = \langle \lambda_j e_j, e_j\rangle = \lambda_j \langle e_j, e_j\rangle = \lambda_j \geq 0$. So all eigenvalues are nonnegative.

Conversely, suppose all eigenvalues are nonnegative. For any $v \in V$, we can write $v = v_1 e_1 + \cdots v_n e_n$. Then

$$\langle Tv, v\rangle = \left\langle \sum_{j=1}^{n} T(v_j e_j), v \right\rangle = \sum_{j=1}^{n} \lambda_j \langle v_j e_j, v\rangle = \sum_{j=1}^{n} \lambda_j \langle v_j e_j, v_j e_j\rangle \geq 0,$$

so T is positive.

8. (a) (Frobenius inequality) If A, B, and C are rectangular matrices such that the product ABC is defined, then

$$\text{rank}(AB) + \text{rank}(BC) \leq \text{rank}(B) + \text{rank}(ABC)$$

(b) In particular, prove that

$$\text{rank}(AB) \leq \min\left\{\text{rank}(A), \text{rank}(B)\right\}.$$

Solution

(a) Let A be m-by-n, B be n-by-p, and C be p-by-q.
We consider $A|_{\text{Range}(B)}$, the restriction of A to the subspace Range(B). We apply the rank theorem to $A|_{\text{Range}(B)}$ and get

$$\text{Rank}(B) = \dim \text{Null}\left(A|_{\text{Range}(B)}\right) + \text{Rank}\left(A|_{\text{Range}(B)}\right).$$

Note that

$$\text{Range}\left(A|_{\text{Range}(B)}\right) = \text{Range}(AB).$$

Therefore

$$\text{Rank}(B) = \dim \text{Null}\left(A|_{\text{Range}(B)}\right) + \text{Rank}\left(AB\right). \tag{1}$$

We now consider $A|_{\text{Range}(BC)}$, the restriction of A to the subspace Range(BC). We apply the rank theorem and follow the same process as above and get:

$$\text{Rank}(BC) = \dim \text{Null}\left(A|_{\text{Range}(BC)}\right) + \text{Rank}\left(ABC\right). \tag{2}$$

Note that

$$\text{Range}(BC) \subset \text{Range}(B),$$

therefore

$$\dim \text{Null}\left(A|_{\text{Range}(BC)}\right) \leq \dim \text{Null}\left(A|_{\text{Range}(B)}\right). \tag{3}$$

Combining Equations 1, 2, and 3 gives the Frobenius inequality.

(b) Let A be m-by-n, B be n-by-p. We set C to be the zero p-by-p matrix. Then the Frobenius inequality applied to the product ABC gives

$$\text{rank}(AB) \leq \text{rank}(B).$$

Now we set C to be the zero m-by-m matrix. Then the Frobenius inequality applied to the product CAB gives

$$\text{rank}(AB) \leq \text{rank}(A).$$

In summary,

$$\text{rank}(AB) \leq \min\left\{\text{rank}(A), \text{rank}(B)\right\}.$$