University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 14, 2013

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

$\boxed{\text{Good luck!}}$

1.	____________	5.	____________
2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________
		Total	____________

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Applied Linear Algebra Preliminary Exam Committee:
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1. Let $A \in \mathcal{M}_n(\mathbb{C})$, and λ be an eigenvalue of A.

   (a) Show that λ^r is eigenvalue of A^r.

   (b) Provide an example showing that the geometric multiplicity of λ^r as eigenvalue of A^r may be strictly higher than the geometric multiplicity of λ as eigenvalue of A.

   (c) Show that A^T has the same eigenvalues as A.

   (d) Show: If A is orthogonal, then $\frac{1}{\lambda}$ is also an eigenvalue of A.

2. Let $\mathfrak{P}$ be the vector space of polynomials over $\mathbb{R}$ with degree at most 2 with inner product

$$\phi(s,t) := \int_{-1}^{1} s(x) \cdot t(x)\ dx.$$

Let

$$\begin{array}{rccc} \mathcal{F}: & \mathfrak{P} & \longrightarrow & \mathfrak{P}, \\ & ax^2 + bx + c & \longmapsto & 2ax + b \end{array}$$

be a linear map (the *differential operator*). Determine the matrices $A_{\mathcal{F}}$ and $A_{\mathcal{F}^*}$ with respect to the basis

(a) $B := \{1, x, x^2\}$,

(b) $B' := \{\frac{1}{2}x^2 - \frac{1}{2}x, x^2 - 1, \frac{1}{2}x^2 + \frac{1}{2}x\}$.

3. Let A be an n-by-n real symmetric positive semidefinite matrix. Let B be an n-by-n real symmetric positive definite matrix.

   (a) Prove that AB have real nonnegative eigenvalues. (Hint: First prove that AB is similar to a symmetric matrix.)

   (b) Prove that
$$\det(A)\det(B) \leq \left(\frac{\operatorname{trace}(AB)}{n}\right)^n$$

4. Let A be a symmetric positive semidefinite matrix. Prove that

   (a)
$$\rho(A) = \sup_{\|x\|_2 \leq 1} \|Ax\|_2 = \sup_{\|x\|_2 \leq 1} x^T A x,$$

   (b)
$$\|A\|_2 \leq \operatorname{trace}(A).$$

5. Let A be an n-by-m matrix and let $(AA^T)^\dagger$ be the pseudoinverse of AA^T.

   (a) Prove that the nullspace of A is orthogonal to the range of A^T.

   (b) Prove that the expression

   $$x = A^T(AA^T)^\dagger Ax + (x - A^T(AA^T)^\dagger Ax)$$

   produces the orthogonal decomposition of $x \in \mathbb{R}^m$ into the sum of a vector from the range of A^T and a vector from the nullspace of A.

6. Let

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Find all subspaces of $\mathbb{R}^4$ which are invariant under the action of A.

(b) Find the spectral radius of A.

7. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Argue that $\sqrt{A}$ is well defined and evaluate it.

8. A, B, C subpsaces. Prove that

$$(\ \ (A \cap B = A + C) \ \ \text{.and.} \ \ (B \cap C = A + B) \ \ ) \Rightarrow (A = B = C)\,.$$