University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 14, 2013

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	____________	5.	____________
2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________
		Total	____________

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Applied Linear Algebra Preliminary Exam Committee:
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1. Let $A \in \mathcal{M}_n(\mathbb{C})$, and λ be an eigenvalue of A.

   (a) Show that λ^r is eigenvalue of A^r.

   (b) Provide an example showing that the geometric multiplicity of λ^r as eigenvalue of A^r may be strictly higher than the geometric multiplicity of λ as eigenvalue of A.

   (c) Show that A^T has the same eigenvalues as A.

   (d) Show: If A is orthogonal, then $\frac{1}{\lambda}$ is also an eigenvalue of A.

Solution

(a) Let x be an eigenvector of A associated with the eigenvalue λ. (In other words, we have $Ax = \lambda x$ with $x \neq 0$.)

$$A^r x \quad = \quad A^{r-1}(Ax) = A^{r-1}(\lambda x) = \lambda A^{r-1}x = ... = \lambda^r x$$

This shows that x is an eigenvector of A^r associated with the eigenvalue λ^r. (Note that $x \neq 0$.) So λ^r is an eigenvalue of A^r.

(b) Consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The geometric multiplicity of the eigenvalue 0 is 1 for A, while the geometric multiplicity of the eigenvalue 0 is 2 for A^2.

(c) That A and A^T have identical eigenvalues follows from the fact that the determinant of a matrix is the determinant of its transpose. (In other words, for any matrix A, $\det(A) = \det(A^T)$.) The characteristic polynomial of A is

$$\det(A - \lambda I) = \det\left((A - \lambda I)^T\right) = \det(A^T - \lambda I),$$

so A and A^T have the same characteristic polynomial, so A and A^T have the same eigenvalues. Note: Clearly the algebraic multiplicities are the same. One can prove that the geometric multiplicities are the same. However, in general, A and A^T do not have the same eigenvectors.

(d) A is orthogonal means that A is (square) real such that $A^T A = AA^T = I$. First we prove that all eigenvalues of an orthogonal matrix have modulus one. Let λ be an eigenvalue of A. Let x be an eigenvector of A associated with the eigenvalue λ. We have $Ax = \lambda x$. If we transpose-conjugate, we get $x^H A^H = \bar{\lambda} x^H$. Multiplying these last two relations, we get: $x^H A^H A x = \lambda\bar{\lambda} x^H x$ Using the fact that A is real, we get $A^H = A^T$, now $A^H A = A^T A = I$. So we get $x^H x = \lambda\bar{\lambda} x^H x$, since $x^H x \neq 0$, (since $x \neq 0$,) we have $\lambda\bar{\lambda} = 1$. In other words, $|\lambda| = 1$. We proved that all eigenvalues of an orthogonal matrix have modulus 1.

We note that, if a complex number have modulus 1, then the inverse and complex conjugate of this number are the same (, since "$\lambda\bar{\lambda} = 1$").

Coming back to our problem, since A is real, if λ is an eigenvalue of A, then $\bar{\lambda}$ is also an eigenvalue of A, Since A is orthogonal, if λ is an eigenvalue of A, then $|\lambda| = 1$, so $\bar{\lambda} = \frac{1}{\lambda}$. We see that if A is orthogonal, if λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is also an eigenvalue of A.

2. Let $\mathfrak{P}$ be the vector space of polynomials over $\mathbb{R}$ with degree at most 2 with inner product

$$\phi(s,t) := \int_{-1}^{1} s(x)\cdot t(x)\ dx.$$

Let

$$\begin{array}{cccc} \mathcal{F}: & \mathfrak{P} & \longrightarrow & \mathfrak{P}, \\ & ax^2+bx+c & \longmapsto & 2ax+b \end{array}$$

be a linear map (the *differential operator*). Determine the matrices $A_{\mathcal{F}}$ and $A_{\mathcal{F}^*}$ with respect to the basis

(a) $B := \{1, x, x^2\}$,

(b) $B' := \{\frac{1}{2}x^2 - \frac{1}{2}x, x^2 - 1, \frac{1}{2}x^2 + \frac{1}{2}x\}$.

Solution

(a-1)

We have $\mathcal{F}(1) = 0$, $\mathcal{F}(x) = 1$, and $\mathcal{F}(x^2) = 2x$, therefore we have

$$A_{\mathcal{F},B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(a-2)

The Gram matrix associated with the basis B and the inner product ϕ is

$$G_B = \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}.$$

The matrix of $\mathcal{F}^*$, the adjoint of $\mathcal{F}$ in the basis B, $A_{\mathcal{F}^*,B}$, is given by the formula:

$$A_{\mathcal{F}^*,B} = G_B^{-1} A_{\mathcal{F},B}^T G_B.$$

This gives

$$A_{\mathcal{F}^*,B} = \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}^T \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}$$

$$A_{\mathcal{F}^*,B} = \frac{3}{8} \begin{pmatrix} 3 & 0 & -5 \\ 0 & 4 & 0 \\ -5 & 0 & 15 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix},$$

$$A_{\mathcal{F}^*,B} = \begin{pmatrix} 0 & -\frac{5}{2} & 0 \\ 3 & 0 & 1 \\ 0 & \frac{15}{2} & 0 \end{pmatrix}.$$

(b-1)
Note: it is clear that $B' := \{\frac{1}{2}x^2 - \frac{1}{2}x, x^2 - 1, \frac{1}{2}x^2 + \frac{1}{2}x\}$ is a basis of $\mathfrak{P}$.
The matrix of B' in B, $M_{B',B}$, is

$$M_{B',B} = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

The matrix of B in B', $M_{B,B'}$, is the inverse of $M_{B',B}$ and is

$$M_{B,B'} = M_{B,B'}^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Now the matrix of $\mathcal{F}$ in B', $A_{\mathcal{F},B'}$, is given by the formula

$$A_{\mathcal{F},B'} = M_{B,B'} * A_{\mathcal{F},B} * M_{B',B},$$

$$A_{\mathcal{F},B'} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

Therefore

$$A_{\mathcal{F},B'} = \begin{pmatrix} -\frac{3}{2} & -2 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 2 & \frac{3}{2} \end{pmatrix}.$$

(b-2)
The matrix of $\mathcal{F}^*$, the adjoint of $\mathcal{F}$ in the basis B', $A_{\mathcal{F}^*,B'}$, is given by the formula:

$$A_{\mathcal{F}^*,B'} = M_{B,B'} * A_{\mathcal{F}^*,B} * M_{B',B}$$

$$A_{\mathcal{F}^*,B'} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{5}{2} & 0 \\ 3 & 0 & 1 \\ 0 & \frac{15}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

Therefore

$$A_{\mathcal{F}^*,B'} = \begin{pmatrix} -3 & 2 & 2 \\ -\frac{5}{4} & 0 & \frac{5}{4} \\ -2 & -2 & 3 \end{pmatrix}.$$

3. Let A be an n-by-n real symmetric positive semidefinite matrix. Let B be an n-by-n real symmetric positive definite matrix.

   (a) Prove that AB have real nonnegative eigenvalues. (Hint: First prove that AB is similar to a symmetric matrix.)

   (b) Prove that

$$\det(A)\det(B) \leq \left(\frac{\operatorname{trace}(AB)}{n}\right)^n$$

Solution

(a) Since B is symmetric positive definite matrix, it has a Cholesky factorization $B = CC^T$, where C is lower triangular with positive elements on the diagonal. (So C is an invertible matrix.) Now, we hvae

$$AB = (C^{-T}C^T)A(CC^T) = (C^T)^{-1}(C^TAC)C^T.$$

Therefore AB is similar to C^TAC, therefore AB and C^TAC have the same eigenvalues.

Since A is symmetric, C^TAC is symmetric as well, so C^TAC has real eigenvalues. Moreover, since C is invertible and A is positive semidefinite, C^TAC is positive semidefinite as well. Therefore C^TAC is real symmetric positive semidefinite, so it has real nonnegative eigenvalues.

We conclude that AB has real nonnegative eigenvalues.

(b) Let λ_i, $i = 1, \ldots, n$, be the n eigenvalues of AB. (Where we repeat eigenvalues according to their algebraic multiplicities.) On the one hand, we note that

$$\det(A)\det(B) = \det(AB) = \prod_{i=1}^{n} \lambda_i.$$

On the other hand,

$$\operatorname{trace}(AB) = \sum_{i=1}^{n} \lambda_i.$$

In part (a), we proved that all eigenvalues λ_i, $i = 1, \ldots, n$, of AB are real nonnegative, therefore we can apply the arithmetic-geometric-mean inequality. The arithmetic-geometric-mean inequality states that the arithmetic mean of a list of nonnegative real numbers is greater than or equal to the geometric mean of the same list. In our context, this gives:

$$\left(\prod_{i=1}^{n} \lambda_i\right)^{\frac{1}{n}} \leq \frac{\sum_{i=1}^{n} \lambda_i}{n}.$$

Using our previous remarks, we get

$$(\det(A)\det(B))^{\frac{1}{n}} \le \frac{\mathrm{trace}(AB)}{n}.$$

Note: Result in (a) is also true if matrix B is symmetric positive semidefinite matrix. (This can be proved by a continuity argument working on $B + \epsilon I$.) In other words, a more general result for (a) is *Given two n-by-n symmetric positive semidefinite matrices, A and B, their product AB has real nonnegative eigenvalues.* So actually result (b) is true when A and B are two n-by-n symmetric positive semidefinite matrices. In other words, we do not need B to be definite.

4. Let A be a symmetric positive semidefinite matrix. Prove that

(a)

$$\rho(A) = \sup_{\|x\|_2 \leq 1} \|Ax\|_2 = \sup_{\|x\|_2 \leq 1} x^T Ax,$$

(b)

$$\|A\|_2 \leq \text{trace}(A).$$

Solution

(a) • $\rho(A)$ is the spectral radius of A and is by definition the largest eigenvalue of A in modulus. Since A is symmetric positive semidefinite, all eigenvalues of A are real nonnegative, and so there is no need for "moduli" or "absolute values" and we can say in a meanningful manner that $\rho(A)$ is the largest eigenvalue of A.

• We call λ_1 the largest eigenvalue of A (so that $\lambda_1 = \rho(A)$) and consider v_1 an eigenvector of A associated with λ_1 such that $\|v_1\|_2 = 1$. We then use the spectral decomposition theorem to write:

$$A = VDV^T,$$

where V is an orthogonal matrix made of an orthogonal basis of eigenvectors of A with v_1 as its first column and D is a real nonnegative diagonal matrix made of the respective eigenvalues of A with λ_1 as entry (1,1).

• Let x be any vector such that $\|x\|_2 \leq 1$, then

$$\|Ax\|_2^2 = x^T A^T Ax = x^T (VDV^T)(VDV^T)x = (V^T x)^T D^2 (V^T x).$$

Let us call $y = V^T x$, we then get

$$\|Ax\|_2^2 = y^T D^2 y = \sum_{i=1}^{n} \lambda_i^2 y_i^2.$$

Since $\lambda_1 \geq \lambda_i$ for all $i = 1$ to n, we can bound with

$$\|Ax\|_2^2 \leq \lambda_1^2 \sum_{i=1}^{n} y_i^2 = \lambda_1 \|y\|_2^2.$$

Finally, since V is orthogonal matrix (so that $V^T V = VV^T = I$) and $y = V^T x$, we have that $\|y\|_2 = \|x\|_2$ and so since $\|x\|_2 \leq 1$ and we have that $\|y\|_2 \leq 1$ and so

$$\|Ax\|_2 \leq \lambda_1.$$

Since x was taken as any vector such that $\|x\|_2 \leq 1$, we can conclude that

$$\sup_{\|x\|_2 \leq 1} \|Ax\|_2 \leq \lambda_1.$$

Now, consider v_1, since $\|Av_1\|_2 = \lambda_1$ and $\|v_1\| = 1$, we can conclude that

$$\sup_{\|x\|_2 \leq 1} \|Ax\|_2 \geq \lambda_1.$$

In fine

$$\sup_{\|x\|_2 \leq 1} \|Ax\|_2 = \lambda_1.$$

- Let x be any vector such that $\|x\|_2 \leq 1$, then

$$x^T Ax = x^T Ax = x^T (VDV^T)x = (V^T x)^T D(V^T x).$$

Let us call $y = V^T x$, we then get

$$x^T Ax = y^T Dy = \sum_{i=1}^{n} \lambda_i y_i^2.$$

Since $\lambda_1 \geq \lambda_i$ for all $i = 1$ to n, we can bound with

$$x^T Ax \leq \lambda_1 \|y\|_2^2.$$

Finally, since V is orthogonal matrix (so that $V^T V = VV^T = I$) and $y = V^T x$, we have that $\|y\|_2 = \|x\|_2$ and so since $\|x\|_2 \leq 1$ and we have that $\|y\|_2 \leq 1$ and so

$$x^T Ax \leq \lambda_1.$$

Since x was taken as any vector such that $\|x\|_2 \leq 1$, we can conclude that

$$\sup_{\|x\|_2 \leq 1} x^T Ax \leq \lambda_1.$$

Now, consider v_1, since $v_1^T Av_1 = \lambda_1$ and $\|v_1\| = 1$, we can conclude that

$$\sup_{\|x\|_2 \leq 1} x^T Ax \geq \lambda_1.$$

In fine

$$\sup_{\|x\|_2 \leq 1} x^T Ax = \lambda_1.$$

(b) The trace of A, trace(A), is by definition the sum of the diagonal elements of A but we know, by theorem, that this is also the sum of the eigenvalues of A:

$$\text{trace}(A) = \sum_{i=1}^{n} \lambda_i.$$

For a symmetric positive semi-definite matrix, we know that the 2-norm, $\|A\|_2$, is the largest eigenvalue in absolue value.

$$\|A\|_2 = \lambda_1.$$

This latter result is actually what part (a) is about. The definition of the 2-norm of a matrix is

$$\|A\|_2 = \sup_{\|x\|_2 \leq 1} \|Ax\|_2.$$

In part (a), we proved that, for a symmetric positive semidefinite matrix,

$$\rho(A) = \sup_{\|x\|_2 \leq 1} \|Ax\|_2$$

so, in part (a), we proved that, for a symmetric positive semidefinite matrix,

$$\rho(A) = \|A\|_2.$$

Anyway, since all eigenvalues of A are nonnegative, (since A is symmetric positive semi-definite matrix,) we have that

$$\lambda_1 \leq \sum_{i=1}^{n} \lambda_i.$$

All this sums up to

$$\|A\|_2 \leq \mathrm{trace}(A).$$

5. Let A be an n-by-m matrix and let $(AA^T)^\dagger$ be the pseudoinverse of AA^T.

   (a) Prove that the nullspace of A is orthogonal to the range of A^T.

   (b) Prove that the expression

$$x = A^T(AA^T)^\dagger Ax + (x - A^T(AA^T)^\dagger Ax)$$

   produces the orthogonal decomposition of $x \in \mathbb{R}^m$ into the sum of a vector from the range of A^T and a vector from the nullspace of A.

Solution

(a) Let x be in Null(A) and y be in Range(A^T), we want to prove that $x^T y = 0$. Since y be in Range(A^T), there exists $z \in \mathbb{R}^n$ such that $y = A^T z$. Now:

$$x^T y = x^T(A^T z) = (Ax)^T z,$$

And, since x is in Null(A), we have that $Ax = 0$, and so

$$x^T y = 0.$$

(b) Let us call

$$y = A^T(AA^T)^\dagger Ax \quad \text{and} \quad z = (x - A^T(AA^T)^\dagger Ax).$$

We want to prove that the decomposition

$$x = y + z$$

produces the orthogonal decomposition of $x \in \mathbb{R}^m$ into the sum of a vector (y) from the range of A^T and a vector (z) from the nullspace of A.
For this, there are four statements to prove/check.
(1) The fact that $x = y + z$ is obvioulsy correct by construction.
(2) The fact that y is in the range of A^T is clear by construction.
(3) We need to prove that z is in the nullspace of A.
(4) If we prove (3), then we can conclude by part (a), that y and z are orthogonal since, in part (a), we proved that the nullspace of A is orthogonal to the range of A^T.

So our question sums up into proving that the vector $z = (x - A^T(AA^T)^\dagger Ax)$ is in the nullspace of A. Then we are done.

Let us consider Az, we see that $Az = (A - (AA^T)(AA^T)^\dagger A)x$. This leads us to consider the operator $(A - (AA^T)(AA^T)^\dagger A)$.
We consider the full SVD decomposition of A: $A = U\Sigma V^T$ where U is n-by-n orthogonal, V is m-by-m orthogonal, and Σ is n-by-m diagonal with singular

values on the diagonal. We assume that Σ has r nonzero singular values on the diagonal. Clearly, r is less than $\min(m, n)$ and there are $\min(m, n) - r$ additional zero singular values on the diagonal. Here what Σ looks like:

$$\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & 0_{r,m-r} \\ & & \sigma_r & \\ & 0_{n-r,r} & & 0_{n-r,m-r} \end{pmatrix}.$$

In the algebra below, we use the following relations:
(R-1) $V^T V = I_m$ and $U^T U = I_n$, (this is the orthogonality of U and V),
(R-2) $(U(\Sigma\Sigma^T)U^T)^\dagger = U(\Sigma\Sigma^T)^\dagger U^T$, this is the standard singular value based formulation of the pseudo-inverse, (that is, for any n-by-m matrix A, we have that if $A = U\Sigma V^T$ singular value decomposition of A, then $A^\dagger = V\Sigma^\dagger U^T$,)
(R-3) $(\Sigma\Sigma^T)(\Sigma\Sigma^T)^\dagger = \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{pmatrix}$. This comes from the fact that $(\Sigma\Sigma^T)$ is a square n-by-n diagonal matrix which looks like

$$\Sigma\Sigma^T = \begin{pmatrix} \sigma_1^2 & & & \\ & \ddots & & 0_{r,n-r} \\ & & \sigma_r^2 & \\ & 0_{n-r,r} & & 0_{n-r,n-r} \end{pmatrix},$$

and so

$$(\Sigma\Sigma^T)^\dagger = \begin{pmatrix} \sigma_1^{-2} & & & \\ & \ddots & & 0_{r,n-r} \\ & & \sigma_r^{-2} & \\ & 0_{n-r,r} & & 0_{n-r,n-r} \end{pmatrix},$$

and so

$$(\Sigma\Sigma^T)(\Sigma\Sigma^T)^\dagger = \begin{pmatrix} 1 & & & \\ & \ddots & & 0_{r,n-r} \\ & & 1 & \\ & 0_{n-r,r} & & 0_{n-r,n-r} \end{pmatrix} = \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{pmatrix}.$$

Back with $A - (AA^T)(AA^T)^\dagger A$:

$$
\begin{aligned}
(A - (AA^T)(AA^T)^\dagger A) &= (U\Sigma V^T) - (U\Sigma V^T)(U\Sigma V^T)^T((U\Sigma V^T)(U\Sigma V^T)^T)^\dagger(U\Sigma V^T), \\
&= (U\Sigma V^T) - (U\Sigma V^T)(V\Sigma^T U^T)((U\Sigma V^T)(V\Sigma^T U^T))^\dagger(U\Sigma V^T), \\
&= (U\Sigma V^T) - (U(\Sigma\Sigma^T)U^T)((U(\Sigma\Sigma^T)U^T))^\dagger(U\Sigma V^T), \\
&= (U\Sigma V^T) - (U(\Sigma\Sigma^T)U^T)(U(\Sigma\Sigma^T)^\dagger U^T)(U\Sigma V^T), \\
&= (U\Sigma V^T) - (U(\Sigma\Sigma^T)(\Sigma\Sigma^T)^\dagger\Sigma V^T), \\
&= (U\Sigma V^T) - (U(\begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r} \end{pmatrix}\Sigma)V^T), \\
&= (U\Sigma V^T) - (U\Sigma V^T), \\
&= 0.
\end{aligned}
$$

So we see that $A - (AA^T)(AA^T)^\dagger A = 0$. Therefore $Az = 0$ and so z is in the nullspace of A.

6. Let

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Find all subspaces of $\mathbb{R}^4$ which are invariant under the action of A.

(b) Find the spectral radius of A.

Solution

(a)

• We first look for the eigenvalues of A.

$$\begin{aligned} \det(A-\lambda I) &= \begin{vmatrix} \frac{1}{2}-\lambda & 0 & \frac{1}{2} & 0 \\ 0 & -\lambda & 1 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3}-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 1-2\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 1 & 0 \\ 1 & 0 & 2-3\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} \\ &= \frac{1}{6}(1-\lambda)\begin{vmatrix} 1-2\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ 1 & 0 & 2-3\lambda \end{vmatrix} = \frac{1}{6}(1-\lambda)(-\lambda)\begin{vmatrix} 1-2\lambda & 1 \\ 1 & 2-3\lambda \end{vmatrix} \\ &= \frac{1}{6}(1-\lambda)(-\lambda)\left((1-2\lambda)(2-3\lambda)-1\right) = \frac{1}{6}(1-\lambda)(-\lambda)(6\lambda^2-7\lambda+1) \\ &= \frac{1}{6}\lambda(6\lambda-1)(\lambda-1)^2 \end{aligned}$$

The eigenvalues of A are 0, $\frac{1}{6}$ and 1.

• Second, we look for eigenspaces of A.

$\boxed{\lambda_1 = 0}$

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$\boxed{\lambda_2 = \frac{1}{6}}$

$$A - \frac{1}{6}I = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{6} & 1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{5}{6} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & -1 & 6 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow v_2 = \begin{pmatrix} -3 \\ 12 \\ 2 \\ 0 \end{pmatrix}.$$

$\boxed{\lambda_3 = 1}$

$$A - I = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 & 0 \\ \frac{1}{3} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \rightsquigarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

• We see that A is diagonalizable and it has two eigenvalues with multiplicty one ($\lambda_1 = 0$ and $\lambda_2 = \frac{1}{6}$) and one eigenvalues with multiplicty two ($\lambda_3 = 1$). (Since A is diagonalizable, the algebraic and geometric multiplicities are the same so we just used the word "multiplicity" without specifying which one.)

• A good idea is to classify the invariant subspaces by dimension.

0. There is of course the invariant subspace of dimension 0 which is simply:

$$\{0\}.$$

It is invariant under the action of any 4-by-4 matrix.

1. The invariant subspaces of dimension 1 of A are:

$$\begin{aligned}
&E_1 = \text{span}(v_1),\\
&E_2 = \text{span}(v_2), \text{ and}\\
&\text{span}(\alpha v_3 + \beta v_4), \text{ for any "non two zeros" } \alpha \text{ and } \beta.
\end{aligned}$$

(Here, $\alpha v_3 + \beta v_4$ represents any eigenvector associated with the eigenvalue 1.)

2. The invariant subspaces of dimension 2 of A are:

$$\begin{aligned}
&\text{span}(v_1) \oplus \text{span}(v_2),\\
&\text{span}(v_1) \oplus \text{span}(\alpha v_3 + \beta v_4), \text{ for any "non two zeros" } \alpha \text{ and } \beta.\\
&\text{span}(v_2) \oplus \text{span}(\alpha v_3 + \beta v_4), \text{ for any "non two zeros" } \alpha \text{ and } \beta.\\
&E_3 = \text{span}(\{v_3, v_4\})
\end{aligned}$$

3. The invariant subspaces of dimension 3 of A are:

$$\begin{aligned}
&\text{span}(v_1) \oplus \text{span}(v_2) \oplus \text{span}(\alpha v_3 + \beta v_4), \text{ for any "non two zeros" } \alpha \text{ and } \beta.\\
&\text{span}(v_1) \oplus \text{span}(\{v_3, v_4\})\\
&\text{span}(v_2) \oplus \text{span}(\{v_3, v_4\})
\end{aligned}$$

4. There is of course the invariant subspace of dimension 4 which is simply:

$$\mathbb{R}^4.$$

It is invariant under the action of any 4-by-4 matrix.

And that is that.

(b)
The spectral radius of A, $\rho(A)$, and is by definition the largest eigenvalue of A in modulus, therefore, for this particular matrix, we have

$$\rho(A) = 1.$$

7. Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Argue that $\sqrt{A}$ is well defined and evaluate it.

Solution

• We note that A is symmetric. We also can see that A has eigenvalue $\lambda_1 = 0$ with geometric multiplicity 2. A (non-orthogonal) basis for the eigenspace associated with the eigenvalue 0 is for example:

$$\{w_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, w_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\}.$$

The second eigenvalue of A is $\lambda_2 = 3$. An eigenvector associated with the eigenvalue 3 is for example:

$$w_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since the eigenvalues of A are nonnegative, we see that A is **symmetric positive semi-definite**.

• We know that a symmetric positive semi-definite matrix has a unique square root. Therefore $\sqrt{A}$ is well defined. This theorem needs to be known by students. The proof is slightly on the hard side but should be known as well. The proof was not asked in this question. For reference, the theorem is given, for example, in Horn and Johnson's "Matrix Analysis" p.405 Theorem 7.2.6, you can also find it in Axler "Linear Algebra Done Right", 2nd Edition, p. 146, Theorem 7.28. We note that we (and Horn and Johnson) call "symmetric positive semi-definite" is what Axler calls "positive".

• We can obtain $\sqrt{A}$ by "eye-balling" it and writing something along the following lines. We see that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix},$$

so

$$\begin{pmatrix} \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \\ \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \\ \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and so

$$\sqrt{A} = \begin{pmatrix} \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \\ \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \\ \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{pmatrix}.$$

• For more complicated symmetric positive semi-definite matrices, it is not possible (for the standard human being) to eye-ball the square root as we did. So it is good to know a method. Since A is symmetric positive semi-definite, we can diagonalize A in orthonormal basis V and we have $A = VDV^T$, with D diagonal with nonnegative entries on the diagonal and V orthogonal. The square root is then given by $A = VD^{1/2}V^T$.

8. A, B, C subpsaces. Prove that

$$(\ \ (A \cap B = A + C)\ \ \text{.and.}\ \ (B \cap C = A + B)\ \ ) \Rightarrow (A = B = C)\,.$$

Solution

It is clear that, (for any subspaces A, B and C,) on the one hand, we have $C \subset A+C$ and, on the other hand, $A \cap B \subset A$. Now, adding to our problem assumption that $A + C = A \cap B$, we get:

$$C \subset A + C = A \cap B \subset A.$$

Therefore

$$C \subset A.$$

It is clear that, (for any subspaces A, B and C,) on the one hand, we have $A \subset A+B$ and, on the other hand, $B \cap C \subset C$. Now, adding to our problem assumption that $A + B = B \cap C$, we get:

$$A \subset A + B = B \cap C \subset C.$$

Therefore

$$A \subset C.$$

So, $C \subset A$ and $A \subset C$, so

$$A = C.$$

It is clear that, (for any subspaces A, B and C,) on the one hand, we have $B \subset A+B$ and, on the other hand, $B \cap C \subset C$. Now, adding to our problem assumption that $A + B = B \cap C$, we get:

$$B \subset A + B = B \cap C \subset C.$$

Therefore

$$B \subset C.$$

It is clear that, (for any subspaces A, B and C,) on the one hand, we have $C \subset A+C$ and, on the other hand, $A \cap B \subset B$. Now, adding to our problem assumption that $A + C = A \cap B$, we get:

$$C \subset A + C = A \cap B \subset B.$$

Therefore

$$A \subset C.$$

So, $B \subset C$ and $C \subset B$, so

$$B = C.$$