University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 8, 2012

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

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Applied Linear Algebra Preliminary Exam Committee:
Steve Billups (Chair), Alexander Engau, Julien Langou.

1. Find an orthogonal basis for the space P_2 of quadratic polynomials with the inner product $\langle f, g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$.

2. A real $n \times n$ matrix A is an isometry if it preserves length: $\|Ax\| = \|x\|$ for all vectors $x \in \mathbb{R}^n$. Show that the following are equivalent.

   (a) A is an isometry (preserves length).

   (b) $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all vectors x, y, so A preserves inner products.

   (c) $A^{-1} = A^*$.

   (d) The columns of A are unit vectors that are mutually orthogonal.

3. Let $p \geq q$. Let A be a real $p \times q$ matrix with rank q. Prove that the QR-decomposition $A = QR$ is unique if R is forced to have positive entries on its main diagonal, Q is $p \times q$ and R is $q \times q$.

4. Let A and B be $n \times n$ complex matrices such that $AB = BA$. Show that if A has n distinct eigenvalues, then A, B, and AB are all diagonalizable.

5. In this problem, $\mathbb{R}$ is the field of real numbers. Let $(u_1, u_2, \ldots, u_m)$ be an orthonormal basis for subspace $W \neq \{0\}$ of the vector space $V = \mathbb{R}^{n\times 1}$ (under the standard inner product), let U be the $n \times m$ matrix defined by $U = [u_1, u_2, \ldots, u_m]$, and let P be the $n \times n$ matrix defined by $A = UU^T$.

   (a) Prove that if v is any given member of V, then among all the vectors w in W, the one which minimizes $\|v - w\|$ is given by $w = \langle v, u_1\rangle\, u_1 + \langle v, u_2\rangle\, u_2 + \ldots + \langle v, u_m\rangle\, u_m$. (The vector w is called the *projection* of v onto W.)

   (b) Prove: For any vector $x \in \mathbb{R}^{n\times 1}$, the projection w of x onto W is given by $w = Px$.

   (c) Prove: P is a projection matrix. (Recall that a matrix $P \in \mathbb{R}^{n\times n}$ is called a *projection matrix* if and only if P is symmetric and idempotent.

   (d) If $V = \mathbb{R}^{3\times 1}$, and $W = \operatorname{span}[(1, 2, 2)^T, (1, 0, 1)^T]$, find the projection matrix P described above and use it to find the projection of $(2, 2, 2)^T$ onto W.

6. Let $V = \mathbb{R}^5$ and let $T \in \mathcal{L}(V)$ be defined by $T(a, b, c, d, e) = (2a, 2b, 2c + d, a + 2d, b + 2e)$.

   (a) (8 points) Find the characteristic and minimal polynomial of T.

   (b) (8 points) Determine a basis of $\mathbb{R}^5$ consisting of eigenvectors and generalized eigenvectors of T.

   (c) (4 points) Find the Jordan form of T with respect to your basis.

7. Suppose that W is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

8. (a) Prove that a normal operator on a finite dimensional complex inner product space with real eigenvalues is self-adjoint.

   (b) Let V be a finite dimensional real inner product space and let $T : V \to V$ be a self-adjoint operator. Is it true that T must have a cube root? Explain. (A cube root of T is an operator $S : V \to V$ such that $S^3 = T$.)

   (a) Let V be a finite dimensional complex inner product space and $T : V \to V$ be a normal operator with real eigenvalues. Let A be the matrix of T in an orthonormal basis. Since T is normal, T is diagonalizable in an orthonormal basis. Therefore there exists a unitary matrix U ($U^H U = I$) such that $A = UDU^H$ with D diagonal. We also know that the eigenvalues of T are real, so D is a real matrix; in particular, this implies $D = D^H$. In this case: $A^H = (UDU^H)^H = U(D^H)U^H = UDU^H = A$.

   (b) T has a cube root. The proof of existence is by construction. Let A be the matrix of T in an orthonormal basis. Since T is a self-adjoint operator, then T is diagonalizable in an orthonormal basis with real eigenvalues. Therefore there exists a unitary matrix U ($U^H U = I$) such that $A = UDU^H$ with D real and diagonal. Define $S = UD^{1/3}U^H$, (the cube root of D is simply the cube root of the diagonal entries,) then it is clear that $S^3 = T$.