University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 8, 2012

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

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Applied Linear Algebra Preliminary Exam Committee:
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1. Find an orthogonal basis for the space P_2 of quadratic polynomials with the inner product $\langle f, g \rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$.

Solution

Two ways.

First way. Take a first nonzero quadratic polynomial, $x(x+1)$, whose value is 0 in -1 and 0, and nonzero in 1; a second polynomial, $(x-1)(x+1)$, whose value is 0 in -1 and 1, and nonzero in 0; and a third polynomial, $x(x-1)$, whose value is 0 in 0 and 1, and nonzero in -1. Then it is easy to see that these three polynomials are orthogonal with respect to the given scalar product. We just need to normalize accordingly. We find:

$$\frac{\sqrt{2}}{2}x(x+1), \quad (x-1)(x+1), \quad \frac{\sqrt{2}}{2}x(x-1).$$

Second way. We can use the Gram-Schmidt process on three linearly independent vectors in P_2, for example: 1, x, and x^2.

2. A real $n \times n$ matrix A is an isometry if it preserves length: $\|Ax\| = \|x\|$ for all vectors $x \in \mathbb{R}^n$. Show that the following are equivalent.

   (a) A is an isometry (preserves length).

   (b) $\langle Ax, Ay \rangle = \langle x, y \rangle$ for all vectors x, y, so A preserves inner products.

   (c) $A^{-1} = A^*$.

   (d) The columns of A are unit vectors that are mutually orthogonal.

   Solution

   (b)⇒(a). Trivial since $\|x\|$ is defined as $\sqrt{\langle x, x \rangle}$. So if an application preserves inner products, it preserves length.

   (a)⇒(b). Assume that A preserves lengths. Let x and $y \in \mathbb{R}^n$. We have $\|A(x+y)\|^2 = \|(x+y)\|^2$. Let us consider $\|A(x+y)\|^2 - \|(x+y)\|^2$. On the one hand this quantity is zero. On the other hand we have: $\|A(x+y)\|^2 - \|(x+y)\|^2 = \langle A(x+y), A(x+y) \rangle - \langle x+y, x+y \rangle = \langle Ax, Ax \rangle + \langle Ax, Ay \rangle + \langle Ay, Ax \rangle + \langle Ay, Ay \rangle - \langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle - \langle y, y \rangle$. We note that $\langle Ax, Ay \rangle = \langle Ay, Ax \rangle$ (symmetry of the inner product) and that $\langle Ay, Ay \rangle = \|Ay\|^2 = \|y\|^2 = \langle y, y \rangle$ (A preserves lengths). All in all, we obtain that $\|A(x+y)\|^2 - \|(x+y)\|^2 = 2\langle Ax, Ay \rangle - 2\langle x, y \rangle$. Setting this to zero implies: $\langle Ax, Ay \rangle = \langle x, y \rangle$. Therefore A preserves inner products.

   We proved that (a)⇔(b).

   (c)⇒(b). Assume $A^{-1} = A^*$. Let x and $y \in \mathbb{R}^n$. $\langle Ax, Ay \rangle = \langle A^*Ax, y \rangle = \langle A^{-1}Ax, y \rangle = \langle x, y \rangle$. So A preserves inner products.

   (b)⇒(d). Assume A preserves inner products. Let a_j be the jth column of A. Then $\langle a_i, a_j \rangle = \langle Ae_i, Ae_j \rangle = \langle e_i, e_j \rangle$. This proves that the columns of A are unit vectors that are mutually orthogonal.

   (d)⇒(c). Assume that the columns of A are unit vectors that are mutually orthogonal. Let a_j be the jth column of A. This means that $\langle a_j, a_j \rangle = 1$ and for $i \neq j$, $\langle a_i, a_j \rangle = 0$. We know that $A^* = A^H$, we know that $(A^H A)_{ij} = a_i^H a_j = \langle a_i, a_j \rangle$, so $A^*A = A^H A = I$. So $A^* = A^{-1}$.

   We proved that (b)⇔(c)⇔(d).

3. Let $p \geq q$. Let A be a real $p \times q$ matrix with rank q. Prove that the QR-decomposition $A = QR$ is unique if R is forced to have positive entries on its main diagonal, Q is $p \times q$ and R is $q \times q$.

   Solution

   Assume that $A = Q_1 R_1$ and $A = Q_2 R_2$ with R_1, R_2 upper triangular with positive entries on the diagonal and $Q_1^T Q_1 = I_q$ and $Q_2^T Q_2 = I_q$.

   We first note that since A is full rank, R_1 and R_2 are invertible. We have $Q_1 R_1 = Q_2 R_2$, multiplying by Q_1^T and R_2^{-1}, this gives

$$R_1 R_2^{-1} = Q_1^T Q_2.$$

   This means that $Q_1^T Q_2$ is upper triangular. Now multiplying by Q_2^T and R_1^{-1}, this gives

$$R_2 R_1^{-1} = Q_2^T Q_1.$$

   This means that $Q_2^T Q_1$ is upper triangular. So $Q_1^T Q_2$ is lower triangular. $Q_1^T Q_2$ is upper and lower triangular. So it is diagonal (and invertible).

   Let us call $D = Q_1^T Q_2$, (from $R_1 R_2^{-1} = Q_1^T Q_2$,) we see that $R_1 = D R_2$. From $Q_1 R_1 = Q_2 R_2$, we see that $Q_1 = Q_2 D^{-1}$. So now $Q_1^T Q_1 = I$ and $Q_2^T Q_2 = I$ give $D^2 = I$. D has therefore ± 1 on the diagonal.

   We come back to the relation $R_1 = D R_2$. Since the diagonal entry of R_1 are given by $(R_1)_{ii} = D_{ii}(R_2)_{ii}$ and that $(R_1)_{ii}$ and $(R_2)_{ii}$ are both positive, and that $D_{ii} = \pm 1$, we see that this implies: $D_{ii} = 1$. Finally $D = I$ and so:

$$Q_1 = Q_2 \quad \text{and} \quad R_1 = R_2.$$

4. Let A and B be $n \times n$ complex matrices such that $AB = BA$. Show that if A has n distinct eigenvalues, then A, B, and AB are all diagonalizable.

Solution

Let $\lambda_1, \ldots, \lambda_n$ be the n distinct eigenvalues of A with corresponding (nonzero) eigenvectors $v_1, \ldots, v_n$. We know that a list of eigenvectors belonging to distinct eigenvalues must be a linearly independent list. Hence $\mathcal{B} = (v_1, \ldots, v_n)$ is a basis of $\mathbb{C}^n$ consisting of eigenvectors of A, so that A is similar to the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_n)$. Then $ABv_i = BAv_i = B(\lambda_i)v_i = \lambda_i(Bv_i)$. So Bv_i belongs to the 1-dimensional eigenspace of A associated with the eigenvalue λ_i. This means that $Bv_i = \mu_i v_i$. Hence the basis $\mathcal{B}$ is also a basis of eigenvectors of B so that v_i is associated with the eigenvalue μ_i (which might be equal to 0). Then clearly AB is similar to the matrix $\text{diag}(\mu_1\lambda_1, \ldots, \mu_n\lambda_n)$.

5. In this problem, $\mathbb{R}$ is the field of real numbers. Let $(u_1, u_2, \ldots, u_m)$ be an orthonormal basis for subspace $W \neq \{0\}$ of the vector space $V = \mathbb{R}^{n\times 1}$ (under the standard inner product), let U be the $n \times m$ matrix defined by $U = [u_1, u_2, \ldots, u_m]$, and let P be the $n \times n$ matrix defined by $A = UU^T$.

   (a) Prove that if v is any given member of V, then among all the vectors w in W, the one which minimizes $\|v - w\|$ is given by $w = \langle v, u_1\rangle\, u_1 + \langle v, u_2\rangle\, u_2 + \ldots + \langle v, u_m\rangle\, u_m$. (The vector w is called the *projection* of v onto W.)

   (b) Prove: For any vector $x \in \mathbb{R}^{n\times 1}$, the projection w of x onto W is given by $w = Px$.

   (c) Prove: P is a projection matrix. (Recall that a matrix $P \in \mathbb{R}^{n\times n}$ is called a *projection matrix* if and only if P is symmetric and idempotent.

   (d) If $V = \mathbb{R}^{3\times 1}$, and $W = \operatorname{span}[(1,2,2)^T, (1,0,1)^T]$, find the projection matrix P described above and use it to find the projection of $(2,2,2)^T$ onto W.

Solution

(a) First it is clear that $w \in W$. Note as well that $v - w \perp W$ since for all $x \in W$,

$$\begin{aligned}\langle v - w, x\rangle &= \langle (v - \langle v, u_1\rangle u_1 - \ldots - \langle v, u_m\rangle u_m)\,, x\rangle \\ &= \langle v, x\rangle - \langle v, u_1\rangle\langle u_1, x\rangle - \ldots - \langle v, u_m\rangle\langle u_m, x\rangle = 0.\end{aligned}$$

The last equality comes from the fact that since $x \in W$, $x = \langle x\, u_1\rangle u_1 + \ldots + \langle x, u_m\rangle u_m$.

Now consider $x \in W$. We define

$$\begin{aligned}\|v - x\|^2 &= \|(v - w) + (w - x)\|^2 \\ &= \|v - w\|^2 + 2(v - w) \bullet (w - x) + \|w - x\|^2\end{aligned}$$

Since $v - w \perp W$ and $w - x \in W$, we have that $(v - w) \bullet (w - x) = 0$, so that

$$\|v - x\|^2 = \|v - w\|^2 + \|w - x\|^2$$

We see that the minimum for $\|v - x\|$ is $\|v - w\|^2$ and is realized when $x = w$.

(b)

$$\begin{aligned}w &= \langle v, u_1\rangle u_1 + \langle v, u_2\rangle u_2 + \ldots + \langle v, u_m\rangle u_m \\ &= u_1(u_1^T v) + u_2(u_2^T v) + \ldots + u_m(u_m^T v) \\ &= (u_1 u_1^T + u_2 u_2^T + \ldots + u_m u_m^T)v \\ &= UU^T v = Pv.\end{aligned}$$

(c) First, $P^T = (UU^T)^T = UU^T = P$, second, $P^2 = (UU^T)^2 = U(U^TU)U^T = UU^T = P$ where we have used the fact that $U^TU = I$.

(d) An orthogonal basis for W is for example

$$(u_1, u_2) = \left(\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix} \right).$$

We get

$$P = UU^T = \begin{pmatrix} 5/9 & -2/9 & 4/9 \\ -2/9 & 8/9 & 2/9 \\ 4/9 & 2/9 & 5/9 \end{pmatrix}.$$

Finally

$$w = Px = \begin{pmatrix} 14/9 \\ 16/9 \\ 22/9 \end{pmatrix}.$$

6. Let $V = \mathbb{R}^5$ and let $T \in \mathcal{L}(V)$ be defined by $T(a,b,c,d,e) = (2a, 2b, 2c+d, a+2d, b+2e)$.

   (a) (8 points) Find the characteristic and minimal polynomial of T.

   (b) (8 points) Determine a basis of $\mathbb{R}^5$ consisting of eigenvectors and generalized eigenvectors of T.

   (c) (4 points) Find the Jordan form of T with respect to your basis.

Solution

The matrix of T in the standard basis $(e_1, e_2, e_3, e_4, e_5)$ is

$$\begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

We can reorder the basis in $(e_3, e_4, e_1, e_5, e_2)$, the matrix of T in this basis is:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

This answers questions (b) and (c). To answer (a), we readily see that the characteristic polynomial of T is $(x-2)^5$ and the minimal polynomial of T is $(x-2)^3$.

7. Suppose that W is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

Solution

First suppose that there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V. Let x and y in V such that $Tx = Ty$. Multiplying by S, this means $STx = STy$, but ST is the identity so $STx = x$ and $STy = y$, so we get $x = y$, which means T is injective.

Now suppose that T is injective. Consider $w_1, \ldots, w_m$ a basis of Range(T). (We use the fact that W is finite dimensional.) Since $w_1, \ldots, w_m$ belongs to Range(T), there exists $v_1, \ldots, v_m$ in V such that $w_1 = Tv_1$, $w_2 = Tv_2$, ... Moreover since T is injective, $v_1, \ldots, v_m$ are linearly independent. Finally since $w_1, \ldots, w_m$ span Range(T), we get that $v_1, \ldots, v_m$ span V. We conclude that $v_1, \ldots, v_m$ is a basis of V. (So V is itself finite dimensional.)

Now we use the incomplete basis theorem to extend $w_1, \ldots, w_m$ with $w_{m+1}, \ldots, w_n$ so as $w_1, \ldots, w_n$ is a basis of W. Now we define $S : W \to V$ (on the basis $w_1, \ldots, w_n$) such that

$$Sw_1 = v_1, \quad Sw_2 = v_2, \quad \ldots, \quad Sw_m = v_m.$$

and

$$Sw_{m+1} = Sw_{m+2} = \ldots = Sw_n = 0.$$

It is clear that $S \in \mathcal{L}(W, V)$ and that ST is the identity map on V.

8. (a) Prove that a normal operator on a finite dimensional complex inner product space with real eigenvalues is self-adjoint.

   (b) Let V be a finite dimensional real inner product space and let $T : V \to V$ be a self-adjoint operator. Is it true that T must have a cube root? Explain. (A cube root of T is an operator $S : V \to V$ such that $S^3 = T$.)

Solution

(a) Let V be a finite dimensional complex inner product space and $T : V \to V$ be a normal operator with real eigenvalues. Let A be the matrix of T in an orthonormal basis. Since T is normal, T is diagonalizable in an orthonormal basis. Therefore there exists a unitary matrix U ($U^H U = I$) such that $A = UDU^H$ with D diagonal. We also know that the eigenvalues of T are real, so D is a real matrix; in particular, this implies $D = D^H$. In this case: $A^H = (UDU^H)^H = U(D^H)U^H = UDU^H = A$.

(b) T has a cube root. The proof of existence is by construction. Let A be the matrix of T in an orthonormal basis. Since T is a self-adjoint operator, then T is diagonalizable in an orthonormal basis with real eigenvalues. Therefore there exists a unitary matrix U ($U^H U = I$) such that $A = UDU^H$ with D real and diagonal. Define $S = UD^{1/3}U^H$, (the cube root of D is simply the cube root of the diagonal entries,) then it is clear that $S^3 = T$.