University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 9, 2012

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Alexander Engau, Andrew Knyazev, Julien Langou (Chair).

1. Let V be a finite-dimensional real vector space. Let W_1 and W_2 be subspaces of V. We define the following operations:

$$(w_1, w_2) + (w_1', w_2') := (w_1 + w_1', w_2 + w_2')$$

and

$$\alpha * (w_1, w_2) := (\alpha w_1, \alpha w_2)$$

for all $(w_1, w_2) \in W_1 \times W_2$ and $(w_1', w_2') \in W_1 \times W_2$ and all $\alpha \in \mathbb{R}$. The set $W_1 \times W_2$ is a vector space with respect to these operations.

   (a) Let $U := \{(u, -u) : u \in W_1 \cap W_2\}$. Prove that U is a subspace of $W_1 \times W_2$. Also prove that U is isomorphic to $W_1 \cap W_2$.

   (b) Define the map $T : W_1 \times W_2 \to W_1 + W_2$ by $T(w_1, w_2) = w_1 + w_2$. Prove that T is a linear transformation.

   (c) Use the above to prove that $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$.

2. Let $E_{ij} \in \mathbb{R}^{n\times n}$ denote the matrix with 1 in entry (i,j) and 0 everywhere else.

   (a) Prove that E_{ii} and E_{jj} are similar for all $1 \leq i, j \leq n$.

   (b) Given $A, B \in \mathbb{R}^{n\times n}$, define $[A,B] := AB - BA$. A matrix $C \in \mathbb{R}^{n\times n}$ is called a *commutator* in $\mathbb{R}^{n\times n}$ if and only if $C = [A,B]$ for some $A, B \in \mathbb{R}^{n\times n}$. Show that $E_{ii} - E_{jj}$ and E_{ij} are commutators in $\mathbb{R}^{n\times n}$ for all $1 \leq i, j \leq n$ with $i \neq j$.

3. We consider a real linear space V of polynomials on $[a, b]$ of degree no larger than 2012 with the scalar product $\langle f, g\rangle := \int_a^b f(t)g(t)dt$. Let a real-valued function $k(s, t)$ be continuous for $s \in [a, b]$ and $t \in [a, b]$. Let us define the linear map $F : V \longrightarrow V$ by

$$f \longmapsto F(f) = g \text{ such that } g(t) := \int_a^b k(s,t)f(s)ds \text{ for all } t \in [a,b].$$

In other words, we have

$$F(f)(t) = \int_a^b k(s,t)f(s)ds, \text{ for all } t \in [a,b].$$

(a) Determine an explicit expression for F^*, the adjoint of F.

(b) Let n be a positive integer. Show that F is normal if $k(s, t) = (s - t)^n$ and determine for which n the linear map F is self-adjoint.

4. We consider two real valued n-by-n matrices A and B such that A is symmetric positive definite and B is anti-symmetric. Prove that $A + B$ is invertible.

5. Let a and $b \in \mathbb{R}$ such that $a \neq b$. Let A a 6-by-6 real valued matrix such that the characteristic polynomial of A is $\chi_A(X) = (X - a)^4(X - b)^2$ and the minimal polynomial of A is $\pi_A(X) = (X - a)^2(X - b)$. Describe all different possible Jordan forms for A.

6. Let A and B be two square matrices such that

$$AB = A^2 + A + I.$$

Show that A and B commute. (Hint: First show that A is invertible.)

7. (a) Let A be a complex Hermitian matrix. Prove that A is positive definite if and only if all the eigenvalues of A are positive.

   (b) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Let $V = \mathbb{R}^3$. We define the map $* : V \times V \to \mathbb{R}$ by $u * v = u^T A v$ for all $u, v \in V$. Prove that $*$ is an inner product on V.

   (c) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for V.

8. For a complex vector $x = [x_1\, x_2]$, we define the function $f(x) = |x_1| + 2|x_2|$.

   (a) Is $f(x)$ a vector norm?

   (b) Is there some scalar product (x, y) such that $(x, x) = f^2(x)$? (Hint: Use the parallelogram identity.)