University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 9, 2012

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

$\boxed{\text{Good luck!}}$

1.	______________	5.	______________
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3.	______________	7.	______________
4.	______________	8.	______________
		Total	______________

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Applied Linear Algebra Preliminary Exam Committee:
Alexander Engau, Andrew Knyazev, Julien Langou (Chair).

1. Let V be a finite-dimensional real vector space. Let W_1 and W_2 be subspaces of V. We define the following operations:

$$(w_1, w_2) + (w_1', w_2') := (w_1 + w_1', w_2 + w_2')$$

and

$$\alpha * (w_1, w_2) := (\alpha w_1, \alpha w_2)$$

for all $(w_1, w_2) \in W_1 \times W_2$ and $(w_1', w_2') \in W_1 \times W_2$ and all $\alpha \in \mathbb{R}$. The set $W_1 \times W_2$ is a vector space with respect to these operations.

(a) Let $U := \{(u, -u) : u \in W_1 \cap W_2\}$. Prove that U is a subspace of $W_1 \times W_2$. Also prove that U is isomorphic to $W_1 \cap W_2$.

(b) Define the map $T : W_1 \times W_2 \to W_1 + W_2$ by $T(w_1, w_2) = w_1 + w_2$. Prove that T is a linear transformation.

(c) Use the above to prove that $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$.

Solution

(a) Let a and b be two vectors in the subset U. Let α and β be two scalars in $\mathbb{R}$. We want to show that $\alpha a + \beta b$ is in U. This will prove that U is a subspace of $W_1 \times W_2$.

Since a is in U, there exists $u \in W_1 \cap W_2$ such that $a = (u, -u)$. Since b is in U, there exists $v \in W_1 \cap W_2$ such that $b = (v, -v)$. Now

$$\begin{aligned} \alpha a + \beta b &= \alpha(u, -u) + \beta(v, -v) \\ &= ((\alpha u + \beta v), -(\alpha u + \beta v)) \\ &= (w, -w) \end{aligned}$$

where we defined w to be $w = (\alpha u + \beta v)$.

We know that (1) u and v are both in $W_1 \cap W_2$ and (2), since W_1 and W_2 are subspaces, $W_1 \cap W_2$ is a subspace as well (standard theorem); from this, we get that w is in $W_1 \cap W_2$ since w is a linear combination of u and v.

As a consequence, $\alpha a + \beta b$ is $(w, -w)$ with $w \in W_1 \cap W_2$, so $\alpha a + \beta b$ is in U. U is a subspace of $W_1 \times W_2$.

We now want to prove that U is isomorphic to $W_1 \cap W_2$. We consider the map, $S : W_1 \cap W_2 \to U$ defined by $S(u) = (u, -u)$. We prove below that S is an isomorphism by proving that S is (1) linear, (2) surjective and (3) injective.

Firstly, S is a linear map. Proof: Let u and v be two vectors in the subset $W_1 \cap W_2$. Let α and β be two scalars in $\mathbb{R}$. Now

$$\begin{aligned} S(\alpha u + \beta v) &= (\alpha u + \beta v, -\alpha u - \beta v) \\ &= \alpha(u, -u) + \beta(v, -v) \\ &= \alpha S(u) + \beta S(v). \end{aligned} \tag{1}$$

So S is linear.

Secondly, S is surjective. Proof: Let a in U, there exists $u \in W_1 \cap W_2$ such that $a = (u, -u)$. Now $S(u) = (u, -u)$ (by definition of S). So $S(u) = a$. We proved that for all a in U, there exists $u \in W_1 \cap W_2$ (see construction) such that $S(u) = a$.

Finally, S is injective. Proof: Let $u \in W_1 \cap W_2$ such that $S(u) = (0, 0)$. This implies $u = 0$. (Since, by definition of S, $S(u) = (u, -u)$.)

The existence of the isomorphism S between $W_1 \cap W_2$ and U proves that these two spaces are isomorphic.

A consequence of the isomorphism between U and $W_1 \cap W_2$ is that

$$\dim(U) = \dim(W_1 \cap W_2).$$

(b) Let a and b be two vectors in the vector space $W_1 \times W_2$. Let α and β be two scalars in $\mathbb{R}$. Since a is in $W_1 \times W_2$, there exists $a_1 \in W_1$ and $a_2 \in W_2$ such that $a = (a_1, a_2)$. Since b is in $W_1 \times W_2$, there exists $b_1 \in W_1$ and $b_2 \in W_2$ such that $b = (b_1, b_2)$. Now

$$\begin{aligned}
T(\alpha a + \beta b) &= T(\alpha(a_1, a_2) + \beta(b_1, b_2)) \\
&= T((\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2)) \\
&= (\alpha a_1 + \beta b_1) + (\alpha a_2 + \beta b_2) \\
&= \alpha(a_1 + a_2) + \beta(b_1 + b_2) \\
&= \alpha T((a_1, a_2)) + \beta T((b_1, b_2)) \\
&= \alpha T(a) + \beta T(b)
\end{aligned}$$

This proves that T is linear.

(c) We use the rank theorem on T. We have

$$\dim(W_1 \times W_2) = \dim(\mathrm{Null}(T)) + \mathrm{Rank}(T).$$

Now, we know that

$$\dim(W_1 \times W_2) = \dim(W_1) + \dim(W_2).$$

We can also easily prove that

$$\mathrm{Null}(T) = U,$$

so that

$$\dim(\mathrm{Null}(T)) = \dim(U) = \dim(W_1 \cap W_2).$$

(The last equality comes from part 1.) Finally

$$\mathrm{Range}(T) = W_1 + W_2.$$

This is because T is (clearly) surjective in $W_1 + W_2$. This implies that

$$\mathrm{Rank}(T) = \dim(W_1 + W_2).$$

Putting all this together, we get

$$\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2.$$

2. Let $E_{ij} \in \mathbb{R}^{n\times n}$ denote the matrix with 1 in entry (i,j) and 0 everywhere else.

   (a) Prove that E_{ii} and E_{jj} are similar for all $1 \le i, j \le n$.

   (b) Given $A, B \in \mathbb{R}^{n\times n}$, define $[A,B] := AB - BA$. A matrix $C \in \mathbb{R}^{n\times n}$ is called a *commutator* in $\mathbb{R}^{n\times n}$ if and only if $C = [A,B]$ for some $A, B \in \mathbb{R}^{n\times n}$. Show that $E_{ii} - E_{jj}$ and E_{ij} are commutators in $\mathbb{R}^{n\times n}$ for all $1 \le i, j \le n$ with $i \neq j$.

Solution

(a) We define the permutation matrix P_{ij} as the identity matrix with row i and row j swapped. (In the $i = j$ case, P_{ij} is the identity matrix.) Now we claim that

$$E_{jj} = P_{ij}E_{ii}P_{ij}.$$

We also know that P_{ij} is invertible and is its own inverse. ($P_{ij}^2 = I$.) So the last relation can actually be rewritten:

$$E_{jj} = P_{ij}E_{ii}P_{ij}^{-1}.$$

This shows that E_{ii} is similar to E_{jj}.

(Another way to answer this question is to note that E_{ii} and E_{jj} are diagonalizable with the same spectrum.)

(b) First,

$$\begin{aligned} E_{ii} - E_{jj} &= E_{ii} - P_{ij}E_{ii}P_{ij} \\ &= (P_{ij}^2)E_{ii} - P_{ij}E_{ii}P_{ij} \\ &= (P_{ij})(P_{ij}E_{ii}) - (P_{ij}E_{ii})(P_{ij}) \\ &= [P_{ij}, P_{ij}E_{ii}]. \end{aligned}$$

This proves that $E_{ii} - E_{jj}$ is a commutator. (It is the commutator $[P_{ij}, P_{ij}E_{ii}]$). (Note we can have $i = j$ here.)

Another way is to prove that $E_{ii} - E_{jj}$ is a commutator is to write $E_{ii} - E_{jj} = E_{ij}E_{ji} - E_{ji}E_{ij} = [E_{ij}, E_{ji}]$.

Second, for $i \neq j$, we note that $E_{ii}E_{ij} = E_{ij}$ and that $E_{ij}E_{ii} = 0$, (the first equality is true for any i and j, the second requires $i \neq j$,) so that

$$E_{ij} = E_{ii}E_{ij} - E_{ij}E_{ii} = [E_{ii}, E_{ij}]$$

This proves that E_{ij} is a commutator. (It is the commutator $[E_{ii}, E_{ij}]$).

3. We consider a real linear space V of polynomials on $[a,b]$ of degree no larger than 2012 with the scalar product $\langle f,g\rangle := \int_a^b f(t)g(t)dt$. Let a real-valued function $k(s,t)$ be continuous for $s\in[a,b]$ and $t\in[a,b]$. Let us define the linear map $F: V \longrightarrow V$ by

$$f \longmapsto F(f) = g \text{ such that } g(t) := \int_a^b k(s,t)f(s)ds \text{ for all } t\in[a,b].$$

In other words, we have

$$F(f)(t) = \int_a^b k(s,t)f(s)ds, \quad \text{for all } t\in[a,b].$$

(a) Determine an explicit expression for F^*, the adjoint of F.

(b) Let n be a positive integer. Show that F is normal if $k(s,t)=(s-t)^n$ and determine for which n the linear map F is self-adjoint.

Solution

(a) First of all, we note that the space V is finite dimensional.
Let f in V and let g in V.
Under the assumption that n is a postive integer, all functions involved in the integration are continuous, and thus Riemann integrable.

$$\langle g, F(f)\rangle = \int_a^b g(t)\left(\int_a^b k(s,t)f(s)ds\right)dt = \int_a^b\int_a^b k(s,t)f(s)g(t)dsdt = \int_a^b\left(\int_a^b k(s,t)g(t)dt\right)f(s)ds.$$

(The integration switch is valid by Fubini's theorem.)
If we define $F^*: V\longrightarrow V$ by

$$g \longmapsto F^*(g) = h \text{ such that } h(s) := \int_a^b k(s,t)g(t)dt \text{ for all } t\in[a,b],$$

we see that for all f in V and for all g in V,

$$\langle g, F(f)\rangle = \langle F^*(g), f\rangle.$$

Therefore F^* is the adjoint of F.

(b) Let f in V, then

$$\begin{aligned}
(FF^*(f))(r) &= \int_a^b\left(\int_a^b (t-s)^n f(s)ds\right)(r-t)^n dt \\
&= \int_a^b\left(\int_a^b (s-t)^n f(s)ds\right)(t-r)^n dt \\
&= (F^*F(f))(r).
\end{aligned}$$

This proves that F is normal.

Now, for n even, $(s-t)^n = (t-s)^n$, so

$$F(f)(t) = \int_a^b (s-t)^n f(s)ds = \int_a^b (t-s)^n f(s)ds = F^*(f)(t).$$

So, for n even, F is self-adjoint.

For n odd, $(s-t)^n = -(t-s)^n$, so

$$F(f)(t) = \int_a^b (s-t)^n f(s)ds = -\int_a^b (t-s)^n f(s)ds = -F^*(f)(t).$$

F is anti-self-adjoint.

Note: for any value of n, F is not the zero operator, so F can not be both self-adjoint and anti-self-adjoint.

Answer: for n even, F is self-adjoint, otherwise it is not self-adjoint.

4. We consider two real valued n-by-n matrices A and B such that A is symmetric positive definite and B is anti-symmetric. Prove that $A + B$ is invertible.

Solution

Since B is anti-symmetric, (which means, by definition, $B^T = -B$,) for all vector x of size n, we have $x^T Bx = (x^T Bx)^T = (Bx)^T x = x^T B^T x = x^T(-B)x = -x^T Bx$, this implies that $x^T Bx = 0$.

Now let x be a n-by-1 vector such that

$$(A + B)x = 0.$$

Then, multiplying on the left by x^T, this implies

$$x^T(A + B)x = x^T Ax + x^T Bx = x^T Ax = 0.$$

Since A is positive definite, $x^T Ax = 0$ implies $x = 0$.

We conclude that $A + B$ is invertible.

5. Let a and $b \in \mathbb{R}$ such that $a \neq b$. Let A a 6-by-6 real valued matrix such that the characteristic polynomial of A is $\chi_A(X) = (X-a)^4(X-b)^2$ and the minimal polynomial of A is $\pi_A(X) = (X-a)^2(X-b)$. Describe all different possible Jordan forms for A.

Solution

The matrix A has exactly two distinct eigenvalues: a and b. (No more.)

Since the minimal polynomial of A has a term in $(X-a)^2$, we deduce that all Jordan blocks associated with the eigenvalue a are either 1-by-1 or 2-by-2. At least one of them is 2-by-2. Since the characteristic polynomial of A has a term in $(X-a)^4$, the total size of all the Jordan blocks associated with the eigenvalue a needs to be 4. We can therefore have

either one 2-by-2 block and two 1-by-1 blocks

or two 2-by-2 blocks

Since the minimal polynomial of A has a term in $(X-b)$, we deduce that all Jordan blocks associated with the eigenvalue b are 1-by-1. Since the characteristic polynomial of A has a term in $(X-b)^2$, the total size of all the Jordan blocks associated with the eigenvalue b needs to be 2. We therefore need to have

two 1-by-1 blocks

We get

one 2-by-2 block and two 1-by-1 blocks for a,
two 1-by-1 blocks for b

$$\begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & b \end{pmatrix}$$

two 2-by-2 blocks for a,
two 1-by-1 blocks for b

$$\begin{pmatrix} a & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & b \end{pmatrix}$$

Of course the Jordan blocks can appear in any order on the diagonal of the Jordan form. (In which case, we would still consider the Jordan form to be the same.)

6. Let A and B be two square matrices such that

$$AB = A^2 + A + I.$$

Show that A and B commute. (Hint: First show that A is invertible.)

Solution

Since

$$AB = A^2 + A + I,$$

we get that

$$A(B - A - I) = I. \tag{2}$$

This means that A is invertible and that $(B - A - I)$ is A^{-1}.

For square matrices, the left-inverse is the right inverse, so that we also have

$$(B - A - I)A = I. \tag{3}$$

We develop Eq.(**??**) and get

$$AB - A^2 - A - I = 0. \tag{4}$$

We develop Eq.(**??**) and get

$$BA - A^2 - A - I = 0. \tag{5}$$

From Eq.(**??**) and Eq.(**??**), we immediately get

$$AB = BA.$$

So A and B commute.

7. (a) Let A be a complex Hermitian matrix. Prove that A is positive definite if and only if all the eigenvalues of A are positive.

(b) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Let $V = \mathbb{R}^3$. We define the map $* : V \times V \to \mathbb{R}$ by $u * v = u^T A v$ for all $u, v \in V$. Prove that $*$ is an inner product on V.

(c) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for V.

Solution

(a) i. Let A be Hermitian positive definite. This means that, for all $x \neq 0$, $x^H A x$ is real positive. Let λ be eigenvalue of A. Let v be an eigenvector of A associated with the eigenvalue λ such that $v^H v = 1.$. Now we see that $v^H A v = v^H(\lambda v) = \lambda(v^H v) = \lambda$. So λ is real positive.

ii. Let A be Hermitian with all eigenvalues positive. Then, since A is Hermitian, A is diagonalizable with an orthonormal basis. So there exists V such that $A = VDV^H$. Let x be a nonzero vector of size n.

$$x^H A x = x^H(VDV^H)x = x^H V D^{1/2} D^{1/2} V^H x = (D^{1/2} V^H x)^H (D^{1/2} V^H x) = \|D^{1/2} V^H x\|^2 > 0.$$

(b) A is symmetric and the eigenvalues of A are 2, 2, and 4 (trivial computation), so the eigenvalues of A are all positive, so A is symmetric positive definite. Therefore $u^T A v$ defines an inner product. (Theorem used: $u^T A v$ defines an inner product if and only if A is symmetric positive definite.)

(c) We apply the Gram-Schmidt process (with the inner product from (b)) to the basis e_1, e_2, e_3 in order to obtain an orthonormal basis for V. (Orthonormal with respect to the inner product from (b).)

i. $e_1^T A e_1 = 2$ so $\|e_1\| = \sqrt{2}$ so $q_1 = [\sqrt{2}/2, 0, 0]$.
ii. We note that $q_1^T A e_2 = 0$ and that $q_1^T A e_3 = 0$
iii. $e_2^T A e_2 = 3$ so $\|e_2\| = \sqrt{3}$ so $q_2 = [0, \sqrt{3}/3, 0]$.
iv. $q_2^T A e_3 = \sqrt{3}$ so $w = e_3 - \sqrt{3} q_2 = [0, 1/3, 1]$.
v. $w^T A w = 8/3$ so $\|w\| = 2\sqrt{6}/3$, so $q_3 = [0, \sqrt{6}/12, \sqrt{6}/4]$.

An orthonormal basis for V is for example

$$q_1 = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ \sqrt{3}/3 \\ 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 \\ \sqrt{6}/12 \\ \sqrt{6}/4 \end{pmatrix}.$$

8. For a complex vector $x = [x_1\, x_2]$, we define the function $f(x) = |x_1| + 2|x_2|$.

   (a) Is $f(x)$ a vector norm?

   (b) Is there some scalar product (x, y) such that $(x, x) = f^2(x)$? (Hint: Use the parallelogram identity.)

Solution

(a) Yes, $f(x)$ is a vector norm. We can check that f satisfies the three properties of a vector norm.

   i. Let x in $\mathbb{C}^2$, let $\lambda \in \mathbb{C}$, then

$$f(\lambda x) = f([\lambda x_1\, \lambda x_2]) = |\lambda x_1| + 2|\lambda x_2| = |\lambda|(|x_1| + 2|x_2|) = |\lambda| f(x).$$

   ii. Let x in $\mathbb{C}^2$, let y in $\mathbb{C}^2$, then

$$f(x+y) = f([x_1+y_1, x_2+y_2]) = |x_1+y_1|+2|x_2+y_2| \leq |x_1|+|y_1|+2|x_2|+2|y_2| = f(x)+f(y).$$

   iii. Let x in $\mathbb{C}^2$, such that $f(x) = 0$, then $|x_1| + 2|x_2| = 0$, since $|x_1| \geq 0$ and $|x_2| \geq 0$, this implies $|x_1| = 0$ and $|x_2| = 0$, so $x_1 = 0$ and $x_2 = 0$ which means $x = 0$.

(b) If we consider the vectors $x = [1 \quad 0]$ and $y = [0 \quad 1]$, we can check that, on the one hand,

$$f(x - y)^2 + f(x + y)^2 = (3)^2 + (3)^2 = 18,$$

and, on the other,

$$2(f(x)^2 + f(y)^2) = 2((1)^2 + (2)^2) = 10.$$

Therefore

$$f(x - y)^2 + f(x + y)^2 \neq 2(f(x)^2 + f(y)^2).$$

We have checked that the parallelogram equality is not true for the vector norm f. As a consequence, this vector norm does not come from a scalar product. There is no scalar product (x, y) such that $(x, x) = f^2(x)$.

We remind that the validity of the parallelogram identity is a necessary and sufficient condition of the existence of a scalar product associated with the given vector norm.