University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 10, 2011

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

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Applied Linear Algebra Preliminary Exam Committee:
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1. Suppose that T is a linear map from V to $\mathbb{F}$ where $\mathbb{F}$ can be either $\mathbb{R}$ or $\mathbb{C}$. Prove that if a vector u in V is not in $\text{null}(T)$, then

$$V = \text{null}(T) \oplus \{\alpha u : \alpha \in \mathbb{F}\}.$$

Solution

(a) Let $x \in V$ such that $x \in \text{null}(T) \cap \text{span}(u)$. Then, since $x \in \text{span}(u)$, there exists $\gamma \in \mathbb{F}$ such that $x = \gamma u$; since $x \in \text{null}(T)$, we have $T(x) = 0$. Combining both gives $T(\gamma u) = 0$. Using the linearity of T gives, $\gamma T(u) = 0$. But, by assumption, u is not in $\text{null}(T)$, so $T(u) \neq 0$. So $\gamma = 0$, so $x = 0$. So

$$\text{null}(T) \cap \text{span}(u) = \{0\}.$$

(b) Let $x \in V$. We decompose x as:

$$x = \left(x - \frac{T(x)}{T(u)}u\right) + \left(\frac{T(x)}{T(u)}u\right).$$

(Note that it is critical here to have $T(u) \neq 0$ to be able to divide by it.) The left-hand side is $\left(x - \frac{T(x)}{T(u)}u\right)$ and belongs to $\text{null}(T)$, since $T\left(x - \frac{T(x)}{T(u)}u\right) = T(x) - \frac{T(x)}{T(u)}T(u) = T(x) - T(x) = 0$. The right-hand side is $\frac{T(x)}{T(u)}u$ and belongs to $\text{span}(u)$. Therefore:

$$\text{null}(T) + \text{span}(u) = V.$$

So we conclude

$$V = \text{null}(T) \oplus \text{span}(u).$$

2. Let A be an n–by–n complex matrix. Define $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$. Prove that A is normal if every eigenvector of H is also an eigenvector of S.

Solution

(a) Two observations.

i. $A = H + S$.

ii. H is Hermitian (since $H = H^*$). Consequently H is diagonalizable in an orthogonal basis, therefore there exists Q, n–by–n unitary matrix, and D_H, n–by–n diagonal matrix, such that

$$H = QD_HQ^*.$$

We note that, since H is Hermitian, its eigenvalues are all real; therefore, D_H is real. We note that, since S is Skew-Hermitian (since $H = -H^*$), S is diagonalizable in an orthogonal basis and all its eigenvalues are purely imaginary. However, these two remarks are not needed in the following.

(b) Since, every eigenvector of H is also an eigenvector of S, for each $i = 1, \dots, n$, we have that, there exists a complex number $d_S^{(i)}$ (the associated eigenvalue) such that

$$Sq^{(i)} = q^{(i)} d_S^{(i)}.$$

(We could prove that $d_S^{(i)}$ is purely imaginary.) Combining this n vector equalities in one matrix equality reads

$$SQ = QD_S,$$

where D_S is n–by–n diagonal matrix made of the $d_S^{(i)}$ on the diagonal. Since Q is unitary, we get

$$S = QD_SQ^*.$$

In other words, we have unitarily diagonalize S in the Q orthogonal basis.

(c) We are ready to conclude:

$$A = H + S = QD_HQ^* + QD_SQ^* = Q(D_H + D_S)Q^*.$$

Therefore A is diagonalizable in an orthogonal basis. Therefore A is normal.

3. Let M be an n-by-n $\{0,1\}$ tournament matrix. That is $M + M^T = J - I$, where J is the matrix of all 1's. Use the following 5 steps to show that $r(M)$ is greater than or equal to $n-1$. (Note: for each step, you can use any of the previous steps, whether you solve them or not). $r(\cdot)$ denotes the rank function.

   (a) (5 pts) Show that if $B^T = -B$ (i.e. B is skew symmetric), then all the eigenvalues of B are pure imaginary or zero. (B is matrix with real coefficient.)

   (b) (2 pts) Show that $M - M^T$ is skew symmetric.

   (c) (5 pts) Let $A = I + M - M^T$. Use (a) and (b) to show that 0 is not an eigenvalue of A and hence A is nonsingular.

   (d) (4 pts) Use that $A = (A - J) + J$ to show that $r(A - J)$ is greater than or equal to $n - 1$.

   (e) (4 pts) Use (d) to show that $r(M^T)$ is greater than or equal to $n-1$. Conclude.

Solution

(a) Let (λ, x) be an eigencouple of B a skew symmetric matrix. Then, $Bx = \lambda x$ (1). If we multiply on the left (1) by x^H, we get $x^H Bx = \lambda x^H x$ (2). Now we transpose-conjugate (1) and get that $x^H B^H = x^H \overline{\lambda}$, we use the fact that $B^H = B^T$ (since B is real) and that $B^T = -B$ (since B is skew symmetric) to get that $x^H(-B) = x^H \overline{\lambda}$, we multiply by x on the right and rearrange to get that $x^H Bx = -\overline{\lambda} x^H x$ (3). Since x is not zero, (2) and (3) imply that $\lambda = -\overline{\lambda}$. Therefore λ is pure imaginary or zero.

(b) $(M - M^T)^T = M^T - (M^T)^T = M^T - M = -(M - M^T)$, this proves that $M - M^T$ is skew symmetric.

(c) The eigenvalues of A are the eigenvalues of $A - I$ shifted by -1. But $A - I = M - M^T$ so, $A - I$ is skew symmetric (see (b)) and all its eigenvalues are pure imaginary or zero. Therefore all the eigenvalues of A are of the form $1 + \lambda . i$ where $\lambda \in \mathbb{R}$. Consequently, none of them is zero and so A is nonsingular.

(d) We know that, for any matrices A and B, $r(A + B) \leq r(A) + r(B)$. In our case, this gives $r(A) \leq r(A - J) + r(J)$, but $r(J) = 1$ and $r(A) = n$, therefore $r(A - J) \geq n - 1$.

(e) Since $A = I + M - M^T$, $A - J = I - J + M - M^T$. But M is such that $M + M^T = J - I$, so $I - J + M - M^T = -2M^T$. From this we get that $A - J = -2M^T$. Therefore (d) proves that $r(2M^T) \geq n - 1$, so that $r(M^T) \geq n - 1$, and so conclude that so is $r(M)$.

4. For each integer $k \geq 0$, let L_k denote the vector space of all polynomials with coefficients in the field $\mathbb{F}$ and of degree less than or equal to k, i.e., let

$$L_k = \{a_0 + a_1 x + \ldots + a_k x^k : a_0, \ldots, a_k \in \mathbb{F}\}.$$

(a) (3 pts) What is the dimension of L_k as a vector space over $\mathbb{F}$? Exhibit a basis for L_k. No justification required.

(b) (5 pts) Show that

$$W = \{f \in L_k : f(0) + f(1) = 0\}$$

is a subspace of L_k.

(c) (6 pts) What is the dimension of W?

(d) (6 pts) Find a basis for W.

Solution

(a) $\dim(L_k) = k + 1$, a basis (for example) is 1, x, x^2, ..., x^k. (Also called the monomial basis.)

(b) Let p and q be two polynomials in W. Let μ and ν be two numbers in $\mathbb{F}$. We have

$$\begin{aligned}(\mu p + \nu q)(0) + (\mu p + \nu q)(1) &= \mu p(0) + \nu q(0) + \mu p(1) + \nu q(1) \\ &= \mu(p(0) + p(1)) + \nu(q(0) + q(1)) \\ &= \mu 0 + \nu 0 = 0.\end{aligned}$$

Therefore $(\mu p + \nu q)$ is in W. Therefore W is a subspace of L_k.

(c) and (d) We claim that $\left\{x^i - \frac{1}{2}, \text{ for } i = 1, \ldots, k\right\}$ is a basis for W.

i. First notice that these k polynomials all belong to W.
ii. Second notice that these k polynomials are linearly independent (since their degrees are all different).
iii. These two observations imply that $\dim(W) \geq k$.
iv. But since the constant polynomial 1 (for example) is not in W, $\dim(W) < \dim(L_k) = k + 1$.
v. Combining the last two items implies $\dim(W) = k$ which answers (c).
vi. Combining (ii) and (v) implies that $\left\{x^i - \frac{1}{2}, \text{ for } i = 1, \ldots, k\right\}$ is a basis for W. This answers (d).

5. Suppose that A is a real, n-by-n symmetric matrix with $A^3 = A^2 + A - I$. Show that A is invertible and in fact A is its own inverse.

Solution

The relation $A^3 = A^2 + A - I$ writes $(A - I)^2(A + I) = 0$. So this means that the eigenvalues of A are 1 and/or -1. But A is symmetric so there is no defective eigenvalue so this means that $(A - I)(A + I) = 0$, this writes $A^2 = I$: A is invertible and in fact A is its own inverse.

6. Let $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ x & 0 & 1 & x+1 \\ 1 & x-1 & 1 & x+1 \\ x & 0 & x & x \end{bmatrix}$, $x \in \mathbb{R}$. What is the rank of A dependent of $x \in \mathbb{R}$.

Solution

We can first "upper triangularize" the given matrix, yielding

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ x & 0 & 1 & x+1 \\ 1 & x-1 & 1 & x+1 \\ x & 0 & x & x \end{bmatrix} \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \end{matrix}$$

$$\xrightarrow{\substack{(1) \\ (2)-x\cdot(1) \\ (3)-(1) \\ (4)-x\cdot(1)}} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & x & 1 & 1 \\ 0 & x & 1 & x \\ 0 & x & x & 0 \end{bmatrix} \begin{matrix} (5) \\ (6) \\ (7) \\ (8) \end{matrix}$$

$$\xrightarrow{\substack{(5) \\ (6) \\ (6)-(7) \\ (6)-(8)}} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & x & 1 & 1 \\ 0 & 0 & 0 & 1-x \\ 0 & 0 & 1-x & 1 \end{bmatrix} = \tilde{A}.$$

The above operations do not affect the matrix rank, so $\text{rank}(A) = \text{rank}(\tilde{A})$. Hence, we conclude that A has full rank 4 whenever $x \in \mathbb{R}\setminus\{0, 1\}$, and rank 3 both if $x = 0$ (as the second and fourth row in $\tilde{A}$ become identical and thus linearly dependent), and if $x = 1$ (as the third row of $\tilde{A}$ reduces to zero).

7. Show that $\det(A_n) = (a + (n-1)b)(a-b)^{n-1}$ where $A_n = \begin{pmatrix} a & b & \cdots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix} \in \mathbb{R}^{n\times n}$ and $a, b \in \mathbb{R}$.

Solution

We can first rewrite the given matrix (determinant-invariant) by subtracting each row (starting from the second) from its predecessor, yielding

$$\det(A_n) = \begin{vmatrix} a & b & \cdots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{vmatrix} = \begin{vmatrix} a & b & \cdots & \cdots & \cdots & b \\ b-a & a-b & 0 & \cdots & \cdots & 0 \\ 0 & b-a & a-b & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 0 & b-a & a-b & 0 \\ 0 & \cdots & \cdots & 0 & b-a & a-b \end{vmatrix} = \det(\tilde{A}_n).$$

We may then develop this determinant from the first row of $\tilde{A}_n$ producing the result to be shown:

$$\begin{aligned} \det(A_n) = \det(\tilde{A}_n) = & \sum_{i=1}^{n} (-1)^{1+i} \tilde{a}_{1i} \det(\tilde{A}_{1i}) \\ = & a(a-b)^{n-1} + \sum_{i=2}^{n} (-1)^{1+i} b(b-a)^{i-1}(a-b)^{n-i} \\ = & a(a-b)^{n-1} + \sum_{i=2}^{n} b(a-b)^{n-1} \\ = & (a + (n-1)b)(a-b)^{n-1}. \end{aligned}$$

Alternate solution: $v_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for A_n associated with the eigenvalue $\lambda_1 = a + (n-1)b$, $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is an eigenvector for A_n associated with the eigenvalue $\lambda_2 = a - b$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}$ is an eigenvector for A_n associated with the eigenvalue $\lambda_2 = a - b$, ... $v_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector for A_n associated with the eigenvalue $\lambda_2 = a - b$. Note that v_1, v_2, ... v_n are linearly independent. Consequently A has eigenvalue $\lambda_1 = a + (n-1)b$ with (geometric and algebraic) multiplicity 1, and $\lambda_2 = a - b$ with (geometric and algebraic) multiplicity $n - 1$. Consequently $\det A_n = \lambda_1 \cdot (\lambda_2)^{n-1} = (a + (n-1)b)(a - b)^{n-1}$.

8. A complex n-by-n matrix P is idempotent if $P^2 = P$. Show that every idempotent matrix is diagonalizable.

Solution

Let P be a complex n-by-n idempotent matrix.

The relation $P^2 = P$ reads as well $P(P - I) = 0$. Therefore the eigenvalues of P are either 0 or 1. We consider the two eigenspaces E_0 (the eigenspace associated with the eigenvalue 0), and E_1 (the eigenspace associated with the eigenvalue 0). Our goal is to prove that $E_0 \oplus E_1 = \mathbb{C}^n$. This will prove that P is diagonalizable.

Note that E_0 is $\text{Null}(P)$.

Note as well that E_1 is $\text{Range}(P)$. This is less obvious. On the one hand, $E_1 \subset \text{Range}(P)$, since, if $x \in E_1$, $x = Tx$ so $x \in \text{Range}(P)$ (in other words an eigenspace is always in the range). On the other hand, if $y \in \text{Range}(P)$, there exists x such that $y = Px$, and so $Py = P^2x = Px = y$ so that $y \in E_1$, so $\text{Range}(P) \subset E_1$,

We now need to prove that $\text{Null}(P) \oplus \text{Range}(P) = \mathbb{C}^n$.

First of, $E_0 \cap E_1 = \{0\}$(, as the intersection of two eigenspaces associated with distinct eigenvalues). So, $\text{Null}(P) \cap \text{Range}(P) = \{0\}$.

So its remains to prove that $\text{Null}(P) + \text{Range}(P) = \mathbb{C}^n$. Let $y \in \mathbb{C}^n$, we can write $y = (Py) + (y - Py)$. The left-hand side (Py) belongs to $\text{Range}(P)$. The right-hand side $(y - Py)$ belongs to $\text{Null}(P)$ since $P(y - Py) = Py - P^2y = Py - Py = 0$.

Therefore $\text{Null}(P) \oplus \text{Range}(P) = \mathbb{C}^n$. So $E_0 \oplus E_1 = \mathbb{C}^n$. This proves that P is diagonalizable.