University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 10, 2011

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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1. Suppose that T is a linear map from V to $\mathbb{F}$ where $\mathbb{F}$ can be either $\mathbb{R}$ or $\mathbb{C}$. Prove that if a vector u in V is not in $\text{null}(T)$, then

$$V = \text{null}(T) \oplus \{\alpha u : \alpha \in \mathbb{F}\}.$$

2. Let A be an n–by–n complex matrix. Define $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$. Prove that A is normal if every eigenvector of H is also an eigenvector of S.

3. Let M be an n-by-n $\{0, 1\}$ tournament matrix. That is $M + M^T = J - I$, where J is the matrix of all 1's. Use the following 5 steps to show that $r(M)$ is greater than or equal to $n-1$. (Note: for each step, you can use any of the previous steps, whether you solve them or not). $r(\cdot)$ denotes the rank function.

   (a) (5 pts) Show that if $B^T = -B$ (i.e. B is skew symmetric), then all the eigenvalues of B are pure imaginary or zero. (B is matrix with real coefficient.)

   (b) (2 pts) Show that $M - M^T$ is skew symmetric.

   (c) (5 pts) Let $A = I + M - M^T$. Use (a) and (b) to show that 0 is not an eigenvalue of A and hence A is nonsingular.

   (d) (4 pts) Use that $A = (A - J) + J$ to show that $r(A - J)$ is greater than or equal to $n - 1$.

   (e) (4 pts) Use (d) to show that $r(M^T)$ is greater than or equal to $n-1$. Conclude.

4. For each integer $k \geq 0$, let L_k denote the vector space of all polynomials with coefficients in the field $\mathbb{F}$ and of degree less than or equal to k, i.e., let

$$L_k = \{a_0 + a_1 x + \ldots + a_k x^k : a_0, \ldots, a_k \in \mathbb{F}\}.$$

(a) (3 pts) What is the dimension of L_k as a vector space over $\mathbb{F}$? Exhibit a basis for L_k. No justification required.

(b) (5 pts) Show that

$$W = \{f \in L_k : f(0) + f(1) = 0\}$$

is a subspace of L_k.

(c) (6 pts) What is the dimension of W?

(d) (6 pts) Find a basis for W.

5. Suppose that A is a real, n-by-n symmetric matrix with $A^3 = A^2 + A - I$. Show that A is invertible and in fact A is its own inverse.

6. Let $A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ x & 0 & 1 & x+1 \\ 1 & x-1 & 1 & x+1 \\ x & 0 & x & x \end{bmatrix}$, $x \in \mathbb{R}$. What is the rank of A dependent of $x \in \mathbb{R}$.

7. Show that $\det(A_n) = (a + (n-1)b)(a-b)^{n-1}$ where $A_n = \begin{pmatrix} a & b & \cdots & b \\ b & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ b & \cdots & b & a \end{pmatrix} \in \mathbb{R}^{n \times n}$ and $a, b \in \mathbb{R}$.

8. A complex n-by-n matrix P is idempotent if $P^2 = P$. Show that every idempotent matrix is diagonalizable.