University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 4, 2010

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	____________	5.	____________
2.	____________	6.	____________
3.	____________	7.	____________
4.	____________	8.	____________
		Total	____________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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1. Let V be a real inner product space with inner product $\langle .,. \rangle$, and suppose that $T \in \mathcal{L}(V)$ is a linear operator $T : V \to V$. Define what an *adjoint* of T is and show that if T has an adjoint, then this adjoint is unique.

2. We consider $\mathcal{M}_n(\mathbb{R})$ the vector space of all n–by–n matrices with real coefficients and supplement it with the inner product $\langle X, Y\rangle \longrightarrow \text{trace}(X^TY)$. Let $A \in \mathcal{M}_n(\mathbb{R})$, and

$$\begin{aligned} \varphi_A : \mathcal{M}_n(\mathbb{R}) &\longrightarrow \mathcal{M}_n(\mathbb{R}) \\ X &\longmapsto A^TXA \end{aligned}$$

Show that $\varphi_A \in \mathcal{L}(\mathcal{M}_n(\mathbb{R}))$ and compute the adjoint of φ_A.

3. (a) Let A be a real symmetric n–by–n matrix. Prove that A is positive definite, i.e., $x^T Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, if and only if all the eigenvalues of A are positive.

(b) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Put $V = \mathbb{R}^3$. Define the map $* : V \times V \to \mathbb{R}$ by $u * v = u^T Av$ for all $u, v \in V$. Prove that $*$ is an inner product on V.

(c) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for V.

4. Let $\mathcal{M}_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real coefficients, and $A \in \mathcal{M}_n(\mathbb{R})$ be diagonalizable. We have a nonsingular matrix W and a diagonal matrix Λ, such that $A = W\Lambda W^{-1}$. Define

$$B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix}.$$

Prove that B is diagonalizable and give the diagonalization of B (i.e. the $2m$ eigencouples of B).
(Hint: one can first consider the $m = 1$ case where $A = 1$.)

5. Let V be a vector space over the real numbers $\mathbb{R}$. Let U_1, U_2, U_3 be subspaces of V.

   (a) Prove that $U_1 \subseteq U_3$ implies that $U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap U_3$ (modular law).

   (b) Give examples to show that none of the following distributive laws holds, in general. $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$ and $U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap (U_1 + U_3)$

6. Let $(u_1, u_2, \ldots, u_m)$ be an orthonormal basis for subspace $W \neq \{0\}$ of the vector space $V = \mathbb{R}^n$ (under the standard inner product), let U be the n–by–m matrix defined by $U = [u_1, u_2, \ldots, u_m]$, and let P be the n–by–n matrix defined by $P = UU^T$.

   (a) Prove that if v is any given member of V, then among all the vectors w in W, the one which minimizes $\|v - w\|$ is given by $w = (v \bullet u_1)u_1 + (v \bullet u_2)u_2 + \ldots + (v \bullet u_m)u_m$ where $v \bullet u$ is the standard inner product. (The vector w is called the *projection* of v onto W.)

   (b) Prove: For any vector $v \in V$, the projection w of v onto W is given by $w = Pv$.

   (c) Prove: P is a projection matrix. (Recall that a matrix $P \in \mathcal{M}_n(\mathbb{R})$ is called a *projection matrix* if and only if P is symmetric ($P^T = P$) and idempotent ($P^2 = P$)).

   (d) If $V = \mathbb{R}^3$, and $W = \text{Span}[(1, 2, 2)^T, (1, 0, 1)^T]$, find the projection matrix P described above and use it to find the projection of $(2, 2, 2)^T$ onto W.

7. Let V be a real inner product space with inner product $\langle .,. \rangle_V$ and let W be a real inner product space with inner product $\langle .,. \rangle_W$ such that $\dim V = \dim W = n < \infty$. Show that there exists a bijective linear mapping $f : V \to W$ so that $\langle x, y \rangle_V = \langle f(x), f(y) \rangle_W$ for all $x, y \in V$.

8. Let n a natural integer, $\mathcal{M}_n(\mathbb{C})$ be the vector space of all $n \times n$ matrices with complex coefficients, and $A = (a_{ij})_{ij} \in \mathcal{M}_n(\mathbb{C})$. Show that

$$\text{Spectrum}(A) \subset \bigcup_{i=1}^{n} \left\{ B'\Big(a_{ii}, \sum_{\substack{1 \,\leq\, j \,\leq\, n \\ j \,\neq\, i}} |a_{ij}|\Big) \right\},$$

where we define for any $a \in \mathbb{C}$ and any $r \in [0, +\infty)$, $B'(a, r)$ by

$$B'(a, r) = \{z \in \mathbb{C}, |z - a| \leq r\}.$$

The $B'\left(a_{ii}, \sum_{1 \leq j \leq n,\ j \neq i} |a_{ij}|\right)$ are called the Gershgorin circles of A.