University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
June 4, 2010

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

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2.	____________	6.	____________
3.	____________	7.	____________
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Applied Linear Algebra Preliminary Exam Committee:
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1. Let V be a real inner product space with inner product $\langle .,. \rangle$, and suppose that $T \in \mathcal{L}(V)$ is a linear operator $T : V \to V$. Define what an *adjoint* of T is and show that if T has an adjoint, then this adjoint is unique.

   Solution

   Let $T \in \mathcal{L}(V)$. An adjoint of T is a linear operator $T^* \in \mathcal{L}(V)$ such that

   $$\forall\, x \in V, \forall\, y \in V, \quad \langle Tx, y \rangle = \langle x, T^* y \rangle.$$

   Claim: The adjoint is unique, if it exists.
   Proof: Let $A \in \mathcal{L}(V)$ be an adjoint of T and let $B \in \mathcal{L}(V)$ be an adjoint of T, then

   $$\forall\, y \in V, \forall\, x \in V, \quad \langle Tx, y \rangle = \langle x, Ay \rangle \text{ and } \langle Tx, y \rangle = \langle x, By \rangle,$$

   so that

   $$\forall\, y \in V, \forall\, x \in V, \quad \langle x, Ay \rangle = \langle x, By \rangle,$$

   using the bilinearity of the inner product,

   $$\forall\, y \in V, \forall\, x \in V, \quad \langle x, Ay - By \rangle = 0,$$

   so that

   $$\forall\, y \in V, \quad (Ay - By) \perp V,$$

   but the only vector in $V^\perp$ is 0, so

   $$\forall\, y \in V, \quad Ay - By = 0,$$

   so that

   $$\forall\, y \in V, \quad Ay = By,$$

   so that

   $$A = B.$$

   Note: The adjoint always exists in finite dimensional inner product spaces. The existence is not necessarily true in infinite dimensional inner product spaces.

2. We consider $\mathcal{M}_n(\mathbb{R})$ the vector space of all n–by–n matrices with real coefficients and supplement it with the inner product $\langle X, Y\rangle \longrightarrow \text{trace}(X^T Y)$. Let $A \in \mathcal{M}_n(\mathbb{R})$, and

$$\begin{array}{rcl} \varphi_A : \mathcal{M}_n(\mathbb{R}) & \longrightarrow & \mathcal{M}_n(\mathbb{R}) \\ X & \longmapsto & A^T X A \end{array}$$

Show that $\varphi_A \in \mathcal{L}(\mathcal{M}_n(\mathbb{R}))$ and compute the adjoint of φ_A.

Solution

We have, $\forall \lambda \in \mathbb{R}, \forall \mu \in \mathbb{R}, \forall X \in \mathcal{M}_n(\mathbb{R}), \forall Y \in \mathcal{M}_n(\mathbb{R})$,

$$\varphi_A(\lambda X+\mu Y) = A^T(\lambda X+\mu Y)A = \lambda(A^T X A)+\mu(A^T Y A) = \lambda\varphi_A(X)+\mu\varphi_A(Y).$$

So $\varphi_A \in \mathcal{L}(\mathcal{M}_n(\mathbb{R}))$.

Let $X \in \mathcal{M}_n(\mathbb{R})$ and let $Y \in \mathcal{M}_n(\mathbb{R})$,

$$\begin{array}{rcl} \langle \varphi_A(X), Y\rangle & = & \text{trace}\left((\varphi_A(X))^T Y\right) \\ & = & \text{trace}\left((A^T X A)^T Y\right) \\ & = & \text{trace}\left(A^T X^T A Y\right), \end{array}$$

we now use the fact that, $\{\ \forall A \in \mathcal{M}_n(\mathbb{R}), \forall B \in \mathcal{M}_n(\mathbb{R}), \text{trace}(AB) = \text{trace}(BA)\}$

$$\begin{array}{rl} = & \text{trace}\left(X^T A Y A^T\right) \\ = & \text{trace}\left(X^T (A Y A^T)\right) \\ = & \text{trace}\left(X^T (\varphi_{A^T}(Y))\right) \\ = & \langle X, \varphi_{A^T}(Y)\rangle \end{array}$$

So

$$(\varphi_A)^* = \varphi_{A^T}.$$

3. (a) Let A be a real symmetric n–by–n matrix. Prove that A is positive definite, i.e., $x^T Ax > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, if and only if all the eigenvalues of A are positive.

(b) Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}$. Put $V = \mathbb{R}^3$. Define the map $* : V \times V \to \mathbb{R}$ by $u * v = u^T Av$ for all $u, v \in V$. Prove that $*$ is an inner product on V.

(c) Use the inner product from above and the Gram-Schmidt orthogonalization process to find an orthonormal basis for V.

Solution

(a) Let A be a real symmetric matrix. We recall that, by definition, A is positive definite if and only $\forall x \in \mathbb{R}^n \backslash \{0\}, x^T Ax > 0$. We also recall that a symmetric matrix is diagonalizable in an orthonormal basis with real eigenvalues, so, for our matrix A, there exists Λ a diagonal n–by–n matrix with real coefficients and V a unitary matrix such that $A = V \Lambda V^T$.

Let A be positive definite. Let λ_i be an eigenvalue of A and v_i a unit-norm eigenvector associated to λ_i, (so that $v_i^T v_i = 1$ and $Av_i = v_i \lambda_i$,) then since A is positive definite, we have $v_i^T A v_i > 0$ which means $\lambda_i > 0$. We have proven that if A is positive definite then all eigenvalues of A are positive. (Alternatively, this direction can be proven by contradiction because otherwise $v_i^T A v_i = \lambda_i v_i^T v_i = \lambda_i \|v_i\|^2 < 0$ and A was not positive definite.)

Let all eigenvalues of A be positive. Let $x \in \mathbb{R}^n \backslash \{0\}$. We have $x^T Ax = x^T V \Lambda V^T x = (V^T x)^T \Lambda (V^T x) = \sum_{i=1}^n \lambda_i (V^T x)_i^2 > 0$. So A is positive definite.

(b) A is symmetric, moreover the eigenvalues of A are 2 and 4 and so are positive, using the previous question, we deduce that A is symmetric positive definite. We check that $*$ satisfies the properties of an inner product on V.

i. $x * y = y * x$,
ii. $(\lambda x) * y = \lambda (x * y)$,
iii. $(x + y) * z = (x * z) + (y * z)$,
iv. $x * x \geq 0$ with equality only for $x = 0$.

(i) comes from the symmetry of A, (ii) and (iii) comes from the linearity of A, (iv) comes from the positive definiteness of A.

(c) We take the elementary basis and use the Gram-Schmidt process on it to obtain on orthonormal basis for V. We obtain

$$q_1 = \begin{pmatrix} \sqrt{2}/2 \\ 0 \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ \sqrt{3}/3 \\ 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 \\ \sqrt{6}/12 \\ \sqrt{6}/4 \end{pmatrix}.$$

4. Let $\mathcal{M}_n(\mathbb{R})$ be the vector space of all $n \times n$ matrices with real coefficients, and $A \in \mathcal{M}_n(\mathbb{R})$ be diagonalizable. We have a nonsingular matrix W and a diagonal matrix Λ, such that $A = W\Lambda W^{-1}$. Define

$$B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix}.$$

Prove that B is diagonalizable and give the diagonalization of B (i.e. the $2m$ eigencouples of B).
(Hint: one can first consider the $m = 1$ case where $A = 1$.)

Solution

Let

$$M = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}.$$

We have

$$p_M(x) = \det(\begin{pmatrix} -x & -1 \\ 2 & 3-x \end{pmatrix}) = x^2 - 3x + 2 = (x-1)(x-2).$$

Since M has two distinct eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 2$, M is diagonalizable.

Then we look for the eigenvectors of M, we find (for example)

$$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

If we call

$$V = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix},$$

we obtain the following diagonalization for M

$$M = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} = VDV^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

Extending this relation to blocks, one can check that

$$B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix} = \begin{pmatrix} I & I \\ -I & -2I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 2A \end{pmatrix} \begin{pmatrix} 2I & I \\ -I & -I \end{pmatrix}.$$

Using the fact that A is diagonalizable, there exists a nonsingular matrix W and a diagonal matrix Λ, such that $A = W\Lambda W^{-1}$. So

$$B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix} = \begin{pmatrix} I & I \\ -I & -2I \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{pmatrix} \begin{pmatrix} W^{-1} & 0 \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} 2I & I \\ -I & -I \end{pmatrix}.$$

which gives the diagonalization of B

$$B = \begin{pmatrix} 0 & -A \\ 2A & 3A \end{pmatrix} = \begin{pmatrix} W & W \\ -W & -2W \end{pmatrix} \begin{pmatrix} \Lambda & 0 \\ 0 & 2\Lambda \end{pmatrix} \begin{pmatrix} 2W^{-1} & W^{-1} \\ -W^{-1} & -W^{-1} \end{pmatrix}.$$

One can check that

$$\begin{pmatrix} W & W \\ -W & -2W \end{pmatrix}^{-1} = \begin{pmatrix} 2W^{-1} & W^{-1} \\ -W^{-1} & -W^{-1} \end{pmatrix}.$$

5. Let V be a vector space over the real numbers $\mathbb{R}$. Let U_1, U_2, U_3 be subspaces of V.

   (a) Prove that $U_1 \subseteq U_3$ implies that $U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap U_3$ (modular law).

   (b) Give examples to show that none of the following distributive laws holds, in general. $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$ and $U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap (U_1 + U_3)$

Solution

(a) Let $U_1 \subseteq U_3$.
One the one hand, we have that $U_1 + (U_2 \cap U_3) \subseteq U_1 + U_2$, on the other, $U_1 + (U_2 \cap U_3) \subseteq U_3$, so that

$$U_1 + (U_2 \cap U_3) \subseteq (U_1 + U_2) \cap U_3.$$

Now let $z \in (U_1 + U_2) \cap U_3$, then there exists $z_1 \in U_1$ and $z_2 \in U_2$ such that $z = z_1 + z_2$ so $z_2 = z - z_1 \in U_3$, so $z_2 \in U_2 \cap U_3$. Therefore $z \in U_1 + (U_2 \cap U_3)$ and so

$$(U_1 + U_2) \cap U_3 \subseteq U_1 + (U_2 \cap U_3).$$

We conclude that

$$(U_1 + U_2) \cap U_3 = U_1 + (U_2 \cap U_3).$$

(b) $U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$ does not hold in general. Consider

$$U_1 = \text{Span}(\begin{pmatrix} 1 \\ 1 \end{pmatrix}), \quad U_2 = \text{Span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}), \quad \text{and} \quad U_3 = \text{Span}(\begin{pmatrix} 0 \\ 1 \end{pmatrix}).$$

Then

$$(U_2 + U_3) = \mathbb{R}^2, \quad U_1 \cap (U_2 + U_3) = \text{Span}(\begin{pmatrix} 1 \\ 1 \end{pmatrix}), \text{ but}$$

$$(U_1 \cap U_2) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}, \quad (U_1 \cap U_3) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}, \quad (U_1 \cap U_2) + (U_1 \cap U_3) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}.$$

$U_1 + (U_2 \cap U_3) = (U_1 + U_2) \cap (U_1 + U_3)$ does not hold in general. Consider again

$$U_1 = \text{Span}(\begin{pmatrix} 1 \\ 1 \end{pmatrix}), \quad U_2 = \text{Span}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}), \quad \text{and} \quad U_3 = \text{Span}(\begin{pmatrix} 0 \\ 1 \end{pmatrix}).$$

Then

$$(U_2 \cap U_3) = \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\}, \quad U_1 + (U_2 \cap U_3) = \text{Span}(\begin{pmatrix} 1 \\ 1 \end{pmatrix}), \text{ but}$$

$$(U_1 + U_2) = \mathbb{R}^2, \quad (U_1 + U_3) = \mathbb{R}^2, \quad (U_1 + U_2) \cap (U_1 + U_3) = \mathbb{R}^2.$$

6. Let $(u_1, u_2, \ldots, u_m)$ be an orthonormal basis for subspace $W \neq \{0\}$ of the vector space $V = \mathbb{R}^n$ (under the standard inner product), let U be the n–by–m matrix defined by $U = [u_1, u_2, \ldots, u_m]$, and let P be the n–by–n matrix defined by $P = UU^T$.

   (a) Prove that if v is any given member of V, then among all the vectors w in W, the one which minimizes $\|v - w\|$ is given by $w = (v \bullet u_1)u_1 + (v \bullet u_2)u_2 + \ldots + (v \bullet u_m)u_m$ where $v \bullet u$ is the standard inner product. (The vector w is called the *projection* of v onto W.)

   (b) Prove: For any vector $v \in V$, the projection w of v onto W is given by $w = Pv$.

   (c) Prove: P is a projection matrix. (Recall that a matrix $P \in \mathcal{M}_n(\mathbb{R})$ is called a *projection matrix* if and only if P is symmetric ($P^T = P$) and idempotent ($P^2 = P$)).

   (d) If $V = \mathbb{R}^3$, and $W = \text{Span}[(1, 2, 2)^T, (1, 0, 1)^T]$, find the projection matrix P described above and use it to find the projection of $(2, 2, 2)^T$ onto W.

Solution

(a) First it is clear that $w \in W$. Note as well that $v - w \perp W$ since for all $x \in W$,

$$\begin{aligned}(v - w \bullet x) &= ((v - (v \bullet u_1)u_1 - \ldots - (v \bullet u_m)u_m), x) \\ &= (v, x) - (v \bullet u_1)(u_1, x) - \ldots - (v \bullet u_m)(u_m, x) = 0.\end{aligned}$$

The last equality comes from the fact that since $x \in W$, $x = (x \bullet u_1)u_1 + \ldots + (x \bullet u_m)u_m$.
Now consider $x \in W$. We define

$$\begin{aligned}\|v - x\|^2 &= \|(v - w) + (w - x)\|^2 \\ &= \|v - w\|^2 + 2(v - w) \bullet (w - x) + \|w - x\|^2\end{aligned}$$

Since $v - w \perp W$ and $w - x \in W$, we have that $(v - w) \bullet (w - x) = 0$, so that

$$\|v - x\|^2 = \|v - w\|^2 + \|w - x\|^2$$

We see that the minimum for $\|v - x\|$ is $\|v - w\|^2$ and is realized when $x = w$.

(b)

$$\begin{aligned}w &= (v \bullet u_1)u_1 + (v \bullet u_2)u_2 + \ldots + (v \bullet u_m)u_m \\ &= u_1(u_1^T v) + u_2(u_2^T v) + \ldots + u_m(u_m^T v) \\ &= (u_1 u_1^T + u_2 u_2^T + \ldots + u_m u_m^T)v \\ &= UU^T v = Pv.\end{aligned}$$

(c) First, $P^T = (UU^T)^T = UU^T = P$, second, $P^2 = (UU^T)^2 = U(U^T U)U^T = UU^T = P$ where we have used the fact that $U^T U = I$.

(d) An orthogonal basis for W is for example

$$(u_1, u_2) = (\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}).$$

We get

$$P = UU^T = \begin{pmatrix} 5/9 & -2/9 & 4/9 \\ -2/9 & 8/9 & 2/9 \\ 4/9 & 2/9 & 5/9 \end{pmatrix}.$$

Finally

$$w = Px = \begin{pmatrix} 14/9 \\ 16/9 \\ 22/9 \end{pmatrix}.$$

7. Let V be a real inner product space with inner product $\langle .,. \rangle_V$ and let W be a real inner product space with inner product $\langle .,. \rangle_W$ such that $\dim V = \dim W = n < \infty$. Show that there exists a bijective linear mapping $f : V \to W$ so that $\langle x, y \rangle_V = \langle f(x), f(y) \rangle_W$ for all $x, y \in V$.

Solution

Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of V and let $\{w_1, \dots, w_n\}$ be an orthonormal basis of W. We define the linear mapping $f : V \to W$ so that

$$\forall i = 1, \dots, n, \quad f(v_i) = w_i.$$

We note that f is correctly and uniquely defined and is bijective.

Claim: f conserves the scalar product (from V to W).

Let $x \in V$, let $y \in V$, then we can decompose x and y onto the orthonormal basis $\{v_1, \dots, v_n\}$ as follows:

$$x = \sum_{i=1}^{n} v_i \langle v_i, x \rangle_V \quad \text{and} \quad y = \sum_{j=1}^{n} v_j \langle v_j, y \rangle_V. \tag{1}$$

We form the inner product $\langle x, y \rangle_V$ and get

$$\langle x, y \rangle_V \quad = \quad \langle \sum_{i=1}^{n} v_i \langle v_i, x \rangle_V, \sum_{j=1}^{n} v_j \langle v_j, y \rangle_V \rangle_V.$$

Using the bilinearity of the inner product $\langle .,. \rangle_V$

$$= \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_i, x \rangle_V \langle v_i, v_j \rangle_V \langle v_j, y \rangle_V,$$

Using the orthonormality of $\{v_1, \dots, v_n\}$, we get

$$= \quad \sum_{i=1}^{n} \langle v_i, x \rangle_V \langle v_i, y \rangle_V,$$

Therefore we have

$$\langle x, y \rangle_V = \sum_{i=1}^{n} \langle v_i, x \rangle_V \langle v_i, y \rangle_V, \tag{2}$$

Back to Equation (1), Applying f and using the linearity of f, we get:

$$f(x) = \sum_{i=1}^{n} f(v_i) \langle v_i, x \rangle_V \quad \text{and} \quad f(y) = \sum_{j=1}^{n} f(v_j) \langle v_j, y \rangle_V.$$

And using the definition of f, we get

$$f(x) = \sum_{i=1}^{n} w_i \langle v_i, x \rangle_V \quad \text{and} \quad f(y) = \sum_{j=1}^{n} w_j \langle v_j, y \rangle_V.$$

We now form the inner product $\langle f(x), f(y) \rangle_W$ and get

$$\langle f(x), f(y) \rangle_W = \langle \sum_{i=1}^{n} w_i \langle v_i, x \rangle_V, \sum_{j=1}^{n} w_j \langle v_j, y \rangle_V \rangle_W.$$

Using the bilinearity of the inner product $\langle ., . \rangle_W$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle v_i, x \rangle_V \langle w_i, w_j \rangle_W \langle v_j, y \rangle_V,$$

Using the orthonormality of $\{w_1, \ldots, w_n\}$, we get

$$= \sum_{i=1}^{n} \langle v_i, x \rangle_V \langle v_i, y \rangle_V,$$

Using Equation (2), we conclude that

$$\langle f(x), f(y) \rangle_W = \langle x, y \rangle_V$$

8. Let n a natural integer, $\mathcal{M}_n(\mathbb{C})$ be the vector space of all $n \times n$ matrices with complex coefficients, and $A = (a_{ij})_{ij} \in \mathcal{M}_n(\mathbb{C})$. Show that

$$\text{Spectrum}(A) \subset \bigcup_{i=1}^{n} \left\{ B'\Big(a_{ii}, \sum_{\substack{1 \le j \le n \\ j \ne i}} |a_{ij}|\Big) \right\},$$

where we define for any $a \in \mathbb{C}$ and any $r \in [0, +\infty)$, $B'(a, r)$ by

$$B'(a, r) = \{z \in \mathbb{C}, |z - a| \le r\}.$$

The $B'\left(a_{ii}, \sum_{1 \le j \le n,\ j \ne i} |a_{ij}|\right)$ are called the Gershgorin circles of A.

Solution

Let $\lambda \in \text{Spectrum}(A)$ and consider an associated eigenvector $x \in \mathbb{R}^n$. (So that $x \ne 0$ and $Ax = x\lambda$.) We write the equality $Ax = x\lambda$ row by row and get

$$\forall\ i = 1, \ldots, n, \quad \sum_{j=1}^{n} a_{ij} x_j = x_i \lambda.$$

Consider i_0 such that

$$|x_{i_0}| = \max_{i=1,\ldots,n} |x_i|.$$

(Note that $|x_{i_0}| \ne 0$ since $x \ne 0$.) Then we get:

$$\begin{aligned} |x_{i_0}(\lambda - a_{i_0 i_0})| &= |\sum_{\substack{1 \le j \le n \\ j \ne i_0}} a_{i_0 j} x_j|, \\ &\le \sum_{\substack{1 \le j \le n \\ j \ne i_0}} |a_{i_0 j}|\ |x_j|, \\ &\le \left(\sum_{\substack{1 \le j \le n \\ j \ne i_0}} |a_{i_0 j}| \right) |x_{i_0}|. \end{aligned}$$

Since $|x_{i_0}| \ne 0$,

$$|\lambda - a_{i_0 i_0}| \le \left(\sum_{\substack{1 \le j \le n \\ j \ne i_0}} |a_{i_0 j}| \right).$$

So

$$\lambda \in \{B' \left(a_{i_0 i_0}, \sum_{\substack{1 \le j \le n \\ j \ne i_0}} |a_{i_0 j}| \right) \}.$$

So

$$\lambda \in \cup_{i=1}^{n}\{B'\left(a_{ii}, \sum_{\substack{1 \le j \le n \\ j \ne i}} |a_{ij}|\right)\}.$$

Since λ was an arbitrary eigenvalue

$$\text{Spectrum}(A) \subset \cup_{i=1}^{n}\{B'\left(a_{ii}, \sum_{\substack{1 \le j \le n \\ j \ne i}} |a_{ij}|\right)\}.$$