University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

15 January 2010, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

PLEASE WRITE ONLY ON ONE SIDE OF EACH SHEET OF PAPER.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless stated otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathbb{R}$ denotes the field of real numbers. $\mathcal{C}^n$ and $\mathbb{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively, written as column vectors. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. For $T \in \mathcal{L}(V)$, the *range* and *null space* of T (sometimes called the *image* and *kernel*) are denoted range (T) and null(T), respectively. $\langle u, v\rangle$ denotes the inner product of vectors u and v. If A is a matrix over a field, then rank (A) is the *rank of* A. For $x \in \mathbb{R}^n$, $\|\cdot\|$ denotes the usual Euclidean norm, unless specified otherwise. If A is an $m \times n$ matrix over a field F, T_A is the linear map defined by

  $$T_A \colon F^n \to F^m \colon x \mapsto Ax.$$

  T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

1. Short answer problems.

   (a) Suppose that A is a normal complex matrix with only one eigenvalue λ. Determine exactly what matrix A must be.

   For the following three parts determine all 2×2 real matrices A for which

   (b) $AA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

   (c) $AA^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

   (d) $AA^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

2. Let A and B be $n \times n$ matrices, V be a finite dimensional vector space, and $T \in \mathcal{L}(V)$. Let row (A) denote the row space of A. Prove or disprove the following statements:

   (a) If $A^2 = 0$, then the rank of A is at most 2.

   (b) If T has no real eigenvalues, then T is invertible.

   (c) $\text{row}\,(AB) \subseteq \text{row}\,(B)$.

   (d) $V = \text{null}(T) \oplus \text{range}\,(T)$.

   (e) There exists a positive integer k so that $V = \text{null}(T^k) \oplus \text{range}\,(T^k)$.

3. Let $T \in \mathcal{L}(\mathbb{R}^n)$ be a normal linear map with $\langle T(x), x\rangle = 0$ for all $x \in \mathbb{R}^n$. Show that $T^* = -T$.

4. Let A be a square matrix over $\mathbb{R}$ and let $\rho(A)$ be the spectral radius of A. Let $\|\cdot\|$ denote the matrix norm induced by the vector norm $\|\cdot\|$. Prove or disprove each of the following:

   (a) $\rho(A) \leq \|A\|$.

   (b) $\rho(AB) \leq \rho(A)\rho(B)$.

   (c) $\rho(A + B) \leq \rho(A) + \rho(B)$.

   (d) If $\|A\| > 1$, then the sequence $\{A^i\}$ diverges as $i \to +\infty$.

5. Let C be an $n \times n$ matrix over the complex numbers.

   (a) Define the terms eigenvalue and eigenvector, and explain what are the algebraic and geometric multiplicities of an eigenvalue.

   (b) Let A be an $m \times n$ complex matrix and let B be an $n \times m$ complex matrix. Let λ be a nonzero eigenvalue of AB with geometric multiplicity equal to k. Show that λ is also an eigenvalue of BA with geometric multiplicity equal to k.

   (c) Explain the connection between the eigenvalues of AB and those of BA, including an example of a case where AB has an eigenvalue that BA does not.

6. Let V be a real inner product space, and $u, v \in V$.

   (a) From the axioms of an inner product space, prove the Cauchy-Schwarz inequality

$$|\langle u, v\rangle| \leq \|u\| \, \|v\| \, .$$

   (b) If $v \neq 0$, show that $\|u+v\| = \|u\| + \|v\|$ if and only if there exists $\alpha \in [0, \infty)$ such that $u = \alpha v$.

7. Jordan Form

   Put $A = \begin{pmatrix} 3 & -1 & 2 & -2 & 2 \\ 0 & 2 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$.

   (a) Determine the Jordan form of A.

   (b) Construct an invertible 5×5 matrix P such that $P^{-1}AP = J$ is in Jordan form.

8. Singular Value Decomposition Let $A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

   (a) Compute a singular value decomposition of A.

   (b) Put $\mathbf{b} = (1, 1, 1)$. Compute the vector $\hat{\mathbf{b}}$ in the row space of A that is closest to $\mathbf{b}$.