University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam With Solutions

15 January 2010, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

PLEASE WRITE ONLY ON ONE SIDE OF EACH SHEET OF PAPER.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless stated otherwise.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathbb{R}$ denotes the field of real numbers. $\mathcal{C}^n$ and $\mathbb{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively, written as column vectors. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. For $T \in \mathcal{L}(V)$, the *range* and *null space* of T (sometimes called the *image* and *kernel*) are denoted range (T) and null(T), respectively. $\langle u, v\rangle$ denotes the inner product of vectors u and v. If A is a matrix over a field, then rank (A) is the *rank of* A. For $x \in \mathbb{R}^n$, $\|\cdot\|$ denotes the usual Euclidean norm, unless specified otherwise. If A is an $m \times n$ matrix over a field F, T_A is the linear map defined by

  $$T_A \colon F^n \to F^m \colon x \mapsto Ax.$$

  T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

1. Short answer problems.

   (a) Suppose that A is a normal complex matrix with only one eigenvalue λ. Determine exactly what matrix A must be.

   For the following three parts determine all 2×2 real matrices A for which

   (b) $AA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

   (c) $AA^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

   (d) $AA^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

   Solution:

   For part (a), since A is a normal complex matrix with λ as its only eigenvalue, it must be unitarily diagonalizable to λI, from which it is clear that $A = \lambda I$.

   For part(b), let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ so that

$$AA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

   which for (b) implies that $b(a+d) = c(a+d) = 1$, so $b = c \neq 0$ and $a^2+bc = a^2+b^2 = 0$ yielding $a = b = 0$, thus showing that such matrix cannot exist. For (c) and (d), we similarly find that

$$AA^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$$

   which immediately yields the same conclusion for (c) because $a^2 + b^2 = c^2 + d^2 = 0$ implies that $a = b = c = d = 0$ so that $ac + bd = 0 \neq 1$ also, and for (d) because $c^2 + d^2 \geq 0 > -1$.

2. Let A and B be $n \times n$ matrices, V be a finite dimensional vector space, and $T \in \mathcal{L}(V)$. Let row (A) denote the row space of A. Prove or disprove the following statements:

   (a) If $A^2 = 0$, then the rank of A is at most 2.

   (b) If T has no real eigenvalues, then T is invertible.

   (c) row $(AB) \subseteq$ row (B).

   (d) $V = \text{null}(T) \oplus \text{range}\,(T)$.

   (e) There exists a positive integer k so that $V = \text{null}(T^k) \oplus \text{range}\,(T^k)$.

   Solution:

(a) If $A^2 = 0$, then the rank of A is at most 2.

False: A counter example is given by

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(b) If T has no real eigenvalues, then T is invertible.

True: 0 is not an eigenvalue of T, so T is invertible.

(c) $\text{row}\,(AB) \subset \text{row}\,(B)$.

True: Suppose $y \in \text{row}\,(AB)$. Then there is a vector c such that $y = c^T AB$. Let $d = A^T c$. Then $y = d^T B$, so $y \in \text{row}\,(B)$.

(d) $V = \text{null}(T) \oplus \text{range}\,(T)$.

False: Let $T = T_A$, where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\text{null}(T) = \text{range}\,(T) = \text{null}(T) \oplus \text{range}\,(T) = \text{span}\,\{(1,0)^T\} \neq V$.

(e) There exists a positive integer k so that $V = \text{null}(T^k) \oplus \text{range}\,(T^k)$.

True: $V \supseteq \text{range}\,(T) \supseteq \text{range}\,(T^2) \supseteq \cdots \supseteq \text{range}\,(T^{n+1}) \supseteq \emptyset$. Thus,

$$n \geq \dim(\text{range}\,(T)) \geq \ldots \geq \dim(\text{range}\,(T^{n+1})) \geq 0.$$

So, for some $k \leq n, \dim(\text{range}\,(T^k)) = \dim(\text{range}\,(T^{k+1}))$. Thus, $\text{range}\,(T^k) = \text{range}\,(T^{k+1}) = T(\text{range}\,(T^k))$, which means that $\text{range}\,(T^k)$ is an invariant subspace of T.

Let $W = \text{range}\,(T^k) \bigcap \text{null}(T^k)$. Then $T^k W = \{0\}$ (since $W \subset \text{null}(T^k)$), and $\dim(T^k W) = \dim(W)$ (since $W \subset \text{range}\,(T^k)$, which is invariant). It follows that $\text{range}\,(T^k) \bigcap \text{null}(T^k) = \{0\}$. Using the fact that for any linear operator S, $\dim(\text{range}\,(S)) + \dim(\text{null}(S)) = n$, we have

$$\begin{aligned} \dim(\text{range}\,(T^k) + \text{null}(T^k)) &= \dim(\text{range}\,(T^k)) + \dim(\text{null}(T^k)) \\ &\quad - \dim\left(\text{range}\,(T^k) \bigcap \dim(\text{null}(T^k)\right) \\ &= \dim(\text{range}\,(T^k)) + \dim(\text{null}(T^k)) = n. \end{aligned}$$

Thus, $\text{range}\,(T^k) + \text{null}(T^k)$ is a subspace of V with dimension n, so is equal to V. Also, since $\text{range}\,(T^k) \bigcap \text{null}(T^k) = \{0\}$, $V = \text{range}\,(T^k) \oplus \text{null}(T^k)$.

3. Let $T \in \mathcal{L}(\mathbb{R}^n)$ be a normal linear map with $\langle T(x), x \rangle = 0$ for all $x \in \mathbb{R}^n$. Show that $T^* = -T$.

Solution: Let $x, y \in \mathbb{R}^n$ be arbitrary, so $x + y \in \mathbb{R}^n$ and

$$\begin{aligned}
0 &= \langle T(x+y), x+y \rangle \\
&= \underbrace{\langle T(x), x \rangle}_{=0} + \langle T(x), y \rangle + \langle T(y), x \rangle + \underbrace{\langle T(y), y \rangle}_{=0} && \text{because } T \text{ is linear} \\
&= \langle T(x), y \rangle + \langle y, T^*(x) \rangle && \text{because } T \text{ is normal} \\
&= \langle T(x), y \rangle + \langle T^*(x), y \rangle && \text{because } T \in \mathcal{L}(\mathbb{R}^n) \\
&= \langle T(x) + T^*(x), y \rangle && \text{again because } T \text{ is linear.}
\end{aligned}$$

Because x and y were chosen arbitrarily, this implies that $T + T^* = 0$ showing that $T^* = -T$.

(NOTE: The assumption that T is normal is not necessary, as shown in the following alternative proof).

Let $T \in \mathcal{L}(\mathbb{R}^n)$ be arbitrary and define $S = T + T^*$ and $K = T - T^*$. Observe that S is Hermitian, that $K^* = -K$ and that, for any x, $\langle K(x), x \rangle = \langle T(x), x \rangle - \langle T^*(x), x \rangle = 0$. It follows that

$$2\langle T(x), x \rangle = \langle S(x), x \rangle + \langle K(x), x \rangle = \langle S(x), x \rangle.$$

Thus, if $\langle T(x), x \rangle = 0$ for all $x \in \mathbb{R}^n$, then $\langle S(x), x \rangle = 0$ for all $x \in \mathbb{R}^n$. Since S is Hermitian, all of its eigenvalues are real. If v is an eigenvector of S associated with eigenvalue λ, then $0 = \langle S(v), v \rangle = \lambda \langle v, v \rangle$, which implies that $\lambda = 0$. Thus, all eigenvalues of S are 0, and since S is Hermitian, $S = 0$.

It follows that $T^* = K^*/2 = -K/2 = -T$.

4. Let A be a square matrix over $\mathbb{R}$ and let $\rho(A)$ be the spectral radius of A. Let $\|\cdot\|$ denote the matrix norm induced by the vector norm $\|\cdot\|$. Prove or disprove each of the following:

   (a) $\rho(A) \leq \|A\|$.

   (b) $\rho(AB) \leq \rho(A)\rho(B)$.

   (c) $\rho(A + B) \leq \rho(A) + \rho(B)$.

   (d) If $\|A\| > 1$, then the sequence $\{A^i\}$ diverges as $i \to +\infty$.

Solution: Part (a). **True.** Let λ_1 denote the eigenvalue with largest magnitude, and let v be an eigenvector associated with λ_1. Then

$$\rho(A) = |\lambda_1| = \frac{\|\lambda_1 v\|}{\|v\|} = \frac{\|Av\|}{\|v\|} \leq \|A\|,$$

where the last inequality follows from the definition of the induced matrix norm.

Part (b). **False.** A counterexample is given by $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Here $\rho(A) = 0$; $\rho(B) = 2$, and $\rho(AB) = 2$.

Part (c) **False.** A counterexample is given by $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Part (d) **False.** A counterexample is given by $A = \begin{pmatrix} .5 & 1 \\ 0 & .5 \end{pmatrix}$. Then $\|A\|_1 = 1.5$ (where $\|\cdot\|_1$ denotes the 1-norm); but $A^3 = \begin{pmatrix} .125 & .75 \\ 0 & .125 \end{pmatrix}$, and $\|A^3\|_1 = .875$. Thus, $\|A^{3i}\|_1 \le .875^i \to 0$, as $i \to +\infty$. It follows that $A^i \to 0$ as $i \to +\infty$.

5. Let C be an $n \times n$ matrix over the complex numbers.

   (a) Define the terms eigenvalue and eigenvector, and explain what are the algebraic and geometric multiplicities of an eigenvalue.

   (b) Let A be an $m \times n$ complex matrix and let B be an $n \times m$ complex matrix. Let λ be a nonzero eigenvalue of AB with geometric multiplicity equal to k. Show that λ is also an eigenvalue of BA with geometric multiplicity equal to k.

   (c) Explain the connection between the eigenvalues of AB and those of BA, including an example of a case where AB has an eigenvalue that BA does not.

   Solution: Given the matrix C as above, if v is a nonzero vector in $\mathbb{C}^n$ for which $Cv = \lambda v$ for some complex number λ, then v is an *eigenvector* of C associated with (or "belonging to") the eigenvalue λ. A complex number λ is an eigenvalue of C if and only if it is a root of the characteristic polynomial $f(x)$ of C. The multiplicity of λ as a root of $f(x)$ is the algebraic multiplicity of λ as an eigenvalue of C. The dimension of the (right) null space of $\lambda I - C$ is the geometric multiplicity of λ as an eigenvalue of C, and the geometric multiplicity of λ is always at least 1 and at most the algebraic multiplicity of λ.

   Now suppose that λ is a nonzero eigenvalue of AB with geometric multiplicity equal to k. This means that there is a linearly independent list $(v_1, \ldots, v_k)$ of eigenvectors of AB belonging to the eigenvalue λ. Then $(ABv_1, \ldots, ABv_k) = (\lambda v_1, \ldots, \lambda v_k)$ is also a linearly independent list, and this easily implies that $(Bv_1, \ldots, Bv_k)$ is a linearly independent list. But then

   $$(BA)Bv_i = B(ABv_i) = B(\lambda v_i) = \lambda(Bv_i),$$

   so that $(Bv_1, \ldots, Bv_k)$ is a linearly independent list of eigenvectors of BA belonging to λ. Hence the geometric multiplicity of λ as an eigenvalue of BA is at least as large as the geometric multiplicity of λ as an eigenvalue of AB. Interchanging the roles of A and B shows that the two geometric multiplicities are the same. **WARNING:** In order for this argument to work we had to know that λ is not zero.

   In general, if λ is a nonzero complex number, then λ is an eigenvalue of AB if and only if it is an eigenvalue of BA. In that case, the algebraic multiplicity of λ is the same for both AB and BA, just as is the case for the geometric multiplicities. However, for $\lambda = 0$ things are different. Suppose that $m > n$. Here AB is $m \times m$ and BA is $n \times n$. The characteristic polynomial of AB is equal to that of BA multiplied by x^{m-n}. So it is possible for AB to have $\lambda = 0$ as an eigenvalue even if BA does not. As a simple example, let $A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and let $B = (1, 1)$. Then

$AB = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $BA = (2)$. So AB has $\lambda = 0$ as an eigenvalue but BA does not.

6. Let V be a real inner product space, and $u, v \in V$.

   (a) From the axioms of an inner product space, prove the Cauchy-Schwarz inequality

   $$|\langle u, v\rangle| \leq \|u\| \, \|v\| \, .$$

   (b) If $v \neq 0$, show that $\|u + v\| = \|u\| + \|v\|$ if and only if there exists $\alpha \in [0, \infty)$ such that $u = \alpha v$.

Solution:

(a) If $v = 0$, then both sides of the inequality are zero, so the inequality holds. Suppose $v \neq 0$, and let

$$w = u - \frac{\langle u, v\rangle}{\|v\|^2} v.$$

Note that w is orthogonal to v, so

$$\|u\|^2 = \left(\frac{\langle u, v\rangle}{\|v\|^2}\right)^2 \|v\|^2 + \|w\|^2 = \frac{\langle u, v\rangle^2}{\|v\|^2} + \|w\|^2 \geq \frac{\langle u, v\rangle^2}{\|v\|^2}. \qquad (1)$$

Mutiplying both sides by $\|v\|^2$ and taking the square root yields the result.

(b)

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v\rangle = \|u\|^2 + 2 \langle u, v\rangle + \|v\|^2 \\ &\leq \|u\|^2 + 2 \|u\| \, \|v\| + \|v\|^2 = (\|u\| + \|v\|)^2 , \end{aligned}$$

with equality if and only if

$$\langle u, v\rangle = \|u\| \, \|v\| \, .$$

In equation 1 above, note that equality holds only if $w = 0$, in which case $u = \pm\alpha v$, where $\alpha = -\frac{\langle u,v\rangle}{\|v\|^2}$. It follows that $\langle u, v\rangle = \|u\| \, \|v\|$ if and only if $u = \alpha v$. Thus, if $\|u + v\| = \|u\| + \|v\|$, then $u = \alpha v$, where α is as defined above, so is nonnegative. Conversely, if $u = \beta v$ for some $\beta \geq 0$, then $\|u + v\| = (\beta + 1) \|v\| = \|\beta v\| + \|v\| = \|u\| + \|v\|$.

7. Jordan Form

Put $A = \begin{pmatrix} 3 & -1 & 2 & -2 & 2 \\ 0 & 2 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$.

(a) Determine the Jordan form of A.

(b) Construct an invertible 5×5 matrix P such that $P^{-1}AP = J$ is in Jordan form.

Solution: Note that A is upper triangular and that the characteristic polynomial of A is $f(x) = |\lambda I - A| = (x-2)^3(x-3)^2$.

Since $\lambda_1 = 3$ has algebraic multiplicity 2, we consider it first. When we row reduce $A - 3I$ we find that the null space of $A - 3I$ has dimension 2 with basis $(e_1, e_4 + e_5)$, where we use the standard notation that $e_1 = (1,0,0,0,0)^T$, $e_2 = (0,1,0,0,0)^T, \ldots, e_5 = (0,0,0,0,1)^T$. This means that the geometric multiplicity of $\lambda_1 = 3$ is the same as its algebraic multiplicity, so the Jordan block associated with $\lambda_1 = 3$ is diagonal.

Now consider $\lambda_2 = 2$. Put $B = A - 2I$. When we row reduce B we find that a basis for its null space is $(e_1 + e_2,\ e_3 + e_4)$. This tells us that the Jordan block associated with $\lambda_2 = 2$ will have one elementary Jordan block of size 2 and one of size 1. So we can now write out the Jordan form of A.

$$\text{The Jordan form of } A \text{ is } J = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Next we compute a basis for the null space of the null space NB^2 of B^2. When

$$\text{we row reduce } B^2 = \begin{pmatrix} 1 & -1 & 2 & -2 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

we find that a basis for the null space of B^2 is $(e_1 + e_2, e_1 - e_3, e_1 + e_4)$. Since the algebraic multiplicity of $\lambda_2 = 2$ is 3, this is as far as we have to go. After playing around with these vectors we easily see the following:

$(e_1 + e_4)$ is a maximal independent list of vectors of NB^2 spanning a space disjoint from NB. Then $B(e_1 + e_4) = -(e_1 + e_2)$ and with the vector $e_3 + e_4$ we have a basis for NB. So we can complete the solution to the problem as follows.

Put $v_1 = e_1$; $v_2 = e_4 + e_5$; $v_3 = -(e_1 + e_2)$; $v_4 = -(e_1 + e_4)$; $v_5 = e_3 + e_4$.

Then if P is the matrix whose columns are the basis vectors $v_1,\ v_2,\ v_3,\ v_4,\ v_5$, we have $P^{-1}AP = J$.

8. Singular Value Decomposition Let $A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

(a) Compute a singular value decomposition of A.

(b) Put $\mathbf{b} = (1,1,1)$. Compute the vector $\hat{\mathbf{b}}$ in the row space of A that is closest to $\mathbf{b}$.

Solution: Since A^TA is 3×3 while AA^T is only 2×2, we begin with computing the eigenvalues of $AA^T = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$. A simple computation shows that $w_1 = (1, -1)$ is an eigenvector of AA^T belonging to eigenvalue $\lambda_1 = 9$. Similarly, $w_2 = (1, 1)$ is an eigenvector of AA^T belonging to eigenvalue $\lambda_2 = 1$. Using Gram-Schmidt on this basis of $\mathbb{R}^2$ (which is trivial since w_1 and w_2 are orthogonal), we find the following orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors of AA^T:

$$\mathcal{B} = \left\{ u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

At this point we know that the (nonzero) singular values of A^T (and hence also of A) are $s_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{1} = 1$. Also we know that the eigenvalues of A^TA must be $\lambda_1 = 9$, $\lambda_2 = 1$, and $\lambda_3 = 0$. At this point we must put $v_i = \frac{1}{s_i} A^T u_i$, for $i = 1, 2$. This gives

$$v_1 = \frac{1}{3\sqrt{2}} \begin{pmatrix} 1 \\ -4 \\ -1 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

According to the general theory, the remaining vector v_3 must be a unit vector that spans the perp of the subspace spanned by v_1 and v_2. It is easy to see that $(2, 1, -2)$ is orthogonal to v_1 and v_2 and has length 3. So we put $v_3 = (2/3, 1/3, -2/3)^T$. This essentially allows us to write down a singular value decomposition of A. Let U be the matrix with columns u_1 and u_2 and let V be the matrix whose columns are the vectors v_1, v_2, v_3. Then

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \text{and } V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/3 & 1 & 2\sqrt{2}/3 \\ -4/3 & 0 & \sqrt{2}/3 \\ -1/3 & 1 & -2\sqrt{2}/3 \end{pmatrix}.$$

Finally, Σ must have the same shape as A and have the singular values down the diagonal, so

$$\Sigma = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Putting this together we find that a singular value decomposition of A is

$$A = U\Sigma V^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1/3 & 1 & 2\sqrt{2}/3 \\ -4/3 & 0 & \sqrt{2}/3 \\ -1/3 & 1 & -2\sqrt{2}/3 \end{pmatrix}^T.$$

This completes a solution to part (a).

Probably the easiest way to do part (b) is to rewrite the problem as: find a least squares solution to $A^T\mathbf{x} = \mathbf{b}$. The usual roles of A and A^T are reversed, but the normal equations give $AA^T\mathbf{x} = A\mathbf{b}$, which becomes

$$\begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

This leads to $\mathbf{x} = (7/9, 11/9)$, from which we find $\hat{\mathbf{b}} = (7/9, 8/9, 11/9)$, which we leave as a row, since the original problem was set that way. One could also have used Gram-Schmidt on the original rows of A to find an orthonormal basis of the row space and use that basis to project $\mathbf{b}$ onto the row space of A.