University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam With Solutions

1 June 2009, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless stated otherwise..
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathbb{R}$ denotes the field of real numbers, and F denotes a field which may be either $\mathcal{C}$ or $\mathbb{R}$. $\mathcal{C}^n$ and $\mathbb{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively, written as column vectors. For $T \in \mathcal{L}(V)$, the *image* (sometimes called the *range*) of T is denoted $\mathrm{Im}(T)$. If A is a matrix over a field, then $\mathrm{rk}(A)$ is the *rank of* A. For $x \in \mathbb{R}^n$, $\|\cdot\|$ denotes the usual Euclidean norm. If A is an $m \times n$ matrix over F, T_A is the linear map defined by

  $$T_A \colon F^n \to F^m \colon x \mapsto Ax.$$

  T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

1. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Define $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$T\colon B \mapsto AB - BA.$$

(i) (8 points) Fix an ordered basis $\mathcal{B}$ of $M_2(\mathbb{R})$ and compute the matrix $[T]_{\mathcal{B}}$ that represents T with respect to this basis.

(ii) (8 points) Compute a basis for each of the eigenspaces of T.

(iii) (4 points) Give the minimal and characteristic polynomials of T and the Jordan form for T.

Solution: We choose the "standard ordered basis"

$$\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$$

where E_{ij} has a 1 in the (i, j) position and zero elsewhere.

Then routine computations show that

$$T\colon E_{11} \mapsto 2(E_{21} - E_{12})$$

$$T\colon E_{12} \mapsto 2(E_{22} - E_{11})$$

$$T\colon E_{21} \mapsto 2(E_{11} - E_{22})$$

$$T\colon E_{22} \mapsto 2(E_{12} - E_{21})$$

From this it is easy to write down he matrix $[T]_{\mathcal{B}}$, and then write down

$$\lambda I - [T]_{\mathcal{B}} = \begin{pmatrix} \lambda & 2 & -2 & 0 \\ 2 & \lambda & 0 & -2 \\ -2 & 0 & \lambda & 2 \\ 0 & -2 & 2 & \lambda \end{pmatrix}.$$

In a few steps we evaluate the determinant of this matrix and find that the characteristic polynomial of T is

$$f(\lambda) = \lambda^2(\lambda - 4)(\lambda + 4).$$

Put $\lambda = 0$ in the matrix and row reduce to find that the null space has the basis $(E_{11} + E_{22},\ E_{12} + E_{21})$

A basis for the eigenspace with $\lambda = 4$ is

$$(E_{11} - E_{12} + E_{21} - E_{22}).$$

A basis for the eigenspace with $\lambda = -4$ is

$$(E_{11} + E_{12} - E_{21} - E_{22}).$$

So the Jordan form is a diagonal matrix with diagonal entries $0, 0, 4, -4$, and the minimal polynomial is

$$p(\lambda) = \lambda(\lambda - 4)(\lambda + 4).$$

2. Let $A \in M_6(\mathcal{C})$ be defined by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Find all of the eigenvalues, eigenvectors, and generalized eigenvectors of A. Construct the characteristic polynomial, the minimal polynomial, and the Jordan form of A.

Solution: The characteristic polynomial is $x^2(x+1)^4$ and the eigenvalues are 0 and -1. The eigenvectors associated with eigenvalue 0 are of the form $[0, a, 0, 0, 0, 0]^T$ and the generalized eigenvectors are of the form $[a, b, 0, 0, 0,]^T$. The eigenvectors associated with -1 are of the form $[0, 0, 0, a, b, -a]^T$, and the generalized eigenvectors are of the form $[0, 0, 0, a, b, c]^T$. The minimal polynomial is $x^2(x+1)^3$. One Jordan form is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

3. Norms of Linear Operators

   (a) Let A be an $m \times n$ real matrix. Prove that there is a real constant M_A such that $\|A\mathbf{x}\| \leq M_A\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.
   Solution: We use two basic facts: First, $\|(x_1, \ldots, x_m)\| \leq \sum_{i=1}^m |x_i|$, by the triangle inequality; Second, recall the Cauchy-Schwartz inequality: $|\mathbf{x} \cdot \mathbf{v}| \leq \|\mathbf{x}\| \cdot \|\mathbf{v}\|$. So we have:

$$\|A\mathbf{v}\| = \|(a_{ij}) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \| = \left\| \begin{matrix} (a_{11}, \ldots, a_{1n}) \cdot \mathbf{v} \\ \vdots \\ (a_{m1}, \ldots, a_{mn}) \cdot \mathbf{v} \end{matrix} \right\| \leq$$

$$\leq \sum_{i=1}^m \|(a_{i1}, \ldots, a_{in}) \cdot \mathbf{v}\| \leq \sum_{i=1}^m \|(a_{i1}, \ldots, a_{in})\| \cdot \|\mathbf{v}\| = M_A\|\mathbf{v}\|,$$

   where $M_A = \sum_{i=1}^m \sqrt{a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2}$.

(b) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Prove that there is some positive constant $\|T\|$ for which

$$\|T(\mathbf{v})\| \leq \|T\| \cdot \|\mathbf{v}\|$$

for all $\mathbf{v} \in \mathbb{R}^n$.

Solution: If $T = T_A : R^n \to R^m$ with $A = (a_{ij})$, then $\|T(\mathbf{v})\| = \|Av\| \leq M_A\|\mathbf{v}\|$ (by part (a)) for all $\mathbf{v} \in R^n$, where $M_A = \sum_{i=1}^m \sqrt{a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2} \in R$.

4. Spheres in Finite Dimensional Real Vector Spaces

Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be an ordered basis of the real vector space V with dimension n. For each $v \in V$ there are unique scalars $c_1, \ldots, c_n \in \mathbb{R}$ for which $v = \sum_{i=1}^n c_i v_i$. Write the *coordinate matrix* $[v]_\mathcal{B}$ *of* v *with respect to the ordered basis* $\mathcal{B}$ as

$$[v]_\mathcal{B} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad \text{so that} \quad v = (v_1, \ldots, v_n) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathcal{B} \cdot [v]_\mathcal{B}.$$

For $\mathbf{c} = [c_1, \ldots, c_n]^T \in \mathbb{R}^n$, we employ the usual Euclidean norm:

$$\|\mathbf{c}\| = \sqrt{\sum_{i=1}^n c_i^2}.$$

For an arbitrary ordered basis $\mathcal{B}$ of V, we define the norm with respect to $\mathcal{B}$ as follows:

$$\|v\|_\mathcal{B} := \|[v]_\mathcal{B}\|.$$

Given the basis $\mathcal{B}$, a specific vector v_0 and a positive number r we can define the n-dimensional sphere with center v_0 and radius r (with respect to $\mathcal{B}$) by

$$S_{r,\mathcal{B}}(v_0) = \{w \in V : \|v_0 - w\|_\mathcal{B} \leq r\}.$$

Problem Let $r > 0$ and let $\mathcal{B}$, $\mathcal{B}'$ be any two ordered bases of V. Show that there is an $r' > 0$ such that

$$S_{r,\mathcal{B}}(\mathbf{0}) \subseteq S_{r',\mathcal{B}'}(\mathbf{0}).$$

Solution: If $\mathcal{B}$ and $\mathcal{B}'$ are two given ordered bases of V, there is an invertible, real $n \times n$ matrix A for which $\mathcal{B}' = \mathcal{B}A$, so that $[v]_\mathcal{B} = A \cdot [v]_{\mathcal{B}'}$. Then there is a constant $\|A\|$ such that $\|AX\| \leq \|A\| \cdot \|X\|$ for any $X \in \mathbb{R}^n$. Hence

$$\|v\|_\mathcal{B} = \|[v]_\mathcal{B}\| = \|A \cdot [v]_{\mathcal{B}'}\| \leq \|A\| \cdot \|[v]_{\mathcal{B}'}\| = \|A\| \cdot \|v\|_{\mathcal{B}'}.$$

From this we see that

$$S_{r,\mathcal{B}A}(\mathbf{0}) \subseteq S_{r\|A\|,\mathcal{B}}(\mathbf{0}). \tag{1}$$

Of course the argument is symmetric in $\mathcal{B}$ and $\mathcal{B}'$.

5. Fredholm Alternative

Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Show that exactly one of the following systems has a solution:

i) $Ax = b$

ii) $A^T y = 0, \quad y^T b \neq 0$.

Note: Our notation is $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, so $y^T = [y_1, \dots, y_m]$.

Solution: If $b \in \mathrm{col}A$, then statement i) has a solution, but since $\mathrm{col}A \perp \mathrm{null}A^T$, statement ii) has no solution.

If $b \notin \mathrm{col}A$, then statement i) does not have a solution. In this case, let $z = \mathrm{proj}_{\mathrm{col}A}\, b$ (the orthogonal projection of b onto the column space of A), and define $y = b - z$. Note that $y \neq 0$ (since $b \notin \mathrm{col}A$). Note also that since z is an orthogonal projection, $y \in (\mathrm{col}A)^{\perp} = \mathrm{null}A^T$. Thus, $A^T y = 0$ and $y^T b = y^T(y + z) = y^T y \neq 0$, so statement ii) has a solution.

6. Upper-triangularization

(a) (12 points) For each of the following, if it is true, merely say so; if it is false, give a counterexample.

(i) If V is a finite-dimensional vector space over $\mathbb{R}$ and $T \in \mathcal{L}(V)$, then V has a basis $\mathcal{B}$ with respect to which $[T]_{\mathcal{B}}$ is upper triangular.

Solution: FALSE Suppose $T \colon \mathbb{R}^2 \to \mathbb{R}^2 \colon (x, y) \mapsto (y, -x)$. Then if $\mathcal{S}$ is the standard ordered basis of $\mathbb{R}^2$, the matrix

$$[T]_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that the characteristic polynomial of T is $x^2 + 1$. This polynomial has no real roots, so T has no real eigenvalues, which would have to lie along the diagonal of $[T]_{\mathcal{B}}$ if V had such a basis.

(ii) If V is a finite-dimensional vector spacee over $\mathcal{C}$ and $T \in \mathcal{L}(V)$, then V has a basis $\mathcal{B}$ with respect to which $[T]_{\mathcal{B}}$ is upper triangular.

Solution: TRUE (This is usually called the Theorem of Schur.)

(iii) If V is a finite-dimensional vector space over $\mathcal{C}$ and $S, T \in \mathcal{L}(V)$, then V has a basis $\mathcal{B}$ for which both $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ are upper triangular.

Solution: False Suppose that with respect to some basis $\mathcal{B}'$, S and T have the following matrices:

$$[S]_{\mathcal{B}'} = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \text{ and } [T]_{\mathcal{B}'} = \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix},$$

with $c \neq d$. Then a basis $\mathcal{B}$ of the desired type would exist if and only if there were an invertible matrix $P = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ for which

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \begin{pmatrix} h & -f \\ -g & e \end{pmatrix} = \begin{pmatrix} * & * \\ ch^2 - g^2 & * \end{pmatrix},$$

with a similar equation holding for the other matrix. It follows that there would have to be an invertible matrix P as above with $g^2 = ch^2$ and $g^2 = dh^2$. If $g = 0$, then $h \neq 0$, implying that $c = 0$ and $d = 0$, contradicting $c \neq d$.

(b) (8 points) Show that a normal, upper triangular matrix must be diagonal.

Solution: We may assume that A is $n \times n$ with entries in $\mathcal{C}$, with $A_{kj} = 0$ if $k > j$. Then

$$\overline{A}_{11} A_{11} = \sum_{k=1}^{n} \overline{A}_{k1} A_{k1} = \sum_{k=1}^{n} A^*_{1k} A_{k1} = (A^*A)_{11} =$$

$$= (AA^*)_{11} = \sum_{k=1}^{n} A_{1k} A^*_{k1} = \sum_{k=1}^{n} A_{1k} \overline{A}_{1k}.$$

It now follows that $A_{12} = A_{13} = \cdots = A_{1n} = 0$. Consider the $(2,2)$ entry.

$$(AA^*)_{22} = \sum_{k=1}^{n} A_{2k} (A^*)_{k2} = \sum_{k=2}^{n} A_{2k} \overline{A}_{2k}.$$

This must also equal

$$(A^*A)_{22} = \sum_{k} (A^*)_{2k} A_{k2} = \sum_{k=1}^{2} (A^*)_{2k} A_{k2} = \overline{A}_{22} A_{22}.$$

It follows that $A_{23} = A_{24} = \cdots = A_{2n}$.
Proceed down the rows to show recursively that in fact A must be diagonal.

7. Tournament Matrices

The matrices of this problem are all $n \times n$ with real entries.

(a) Show that if the matrix A is skew-symmetric then $I + A$ is nonsingular.

(b) Show that for arbitrary matrices A and B, $\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)$.

(c) If A is arbitrary and J is the matrix of all 1's, then show that

$$\text{rk}(A - J) \geq \text{rk}(A) - \text{rk}(J).$$

(d) If M is a $(0,1)$-matrix with zeros on the main diagonal and with $M_{ij} = 0$ if and only if $M_{ji} = 1$, show that $\text{rk}(M) \geq n - 1$. (Such a matrix is called a *tournament* matrix.)

Solution: Suppose $A^T = -A$ and that X is a column vector for which $(I + A)X = 0$. Then $AX = -X$ implies that $X = (-A)X = A^T X$, so $X^T A = X^T$. Then $X^T X = (X^T A)X = X^T(AX) = X^T(-X) = -XTX$, which implies that $X^T X = 0$, and hence $X = 0$. So 0 is not an eigenvalue of $I + A$. For the second part, observe that the union of a maximal independent set of rows of A with a maximal independent set of rows of B will certainly span the row space of $A + B$. For the third part, apply the second part to the matrix $A = (A - J) + J$. For the last part, let M be a tournament matrix of order n. Then $M + M^T = J - I$, i.e., $J = I + M + M^T$. Clearly $M - M^T$ is skew-symmetric, so $A = I + M - M^T$ is nonsingular by the first part. Hence $\text{rk}(A) = n$. Then $\text{rk}(A - J) \geq \text{rk}(A) - \text{rk}(J) = n - 1$. But $A - J = -2M^T$, so $\text{rk}(M) = \text{rk}(M^T) = \text{rk}(A - J) \geq n - 1$.

8. Given an $m \times n$ matrix A, the *pseudoinverse of* A, denoted A^+, can be defined as the matrix such that for all $b \in \mathcal{C}^m$, $x^+ := A^+b$ is the least squares solution to the equation $Ax = b$ that has the smallest norm.

   (a) Using the above definition, explain why AA^+ and A^+A must be projection matrices (and are therefore Hermitian). Onto what fundamental subspaces do these matrices project?

   (b) Prove that $AA^+A = A$ and $A^+AA^+ = A^+$. (Note: these two properties, together with the Hermitian properties in part (a) uniquely determine the pseudoinverse).

   (c) If Σ is a real diagonal matrix, what is Σ^+?

   (d) Give an explicit formula for A^+ in terms of the singular value decomposition $A = V\Sigma W^*$. Justify your answer.

Solution:

(a) For x^+ to be a least squares solution to $Ax = b$, Ax^+ must be the orthogonal projection of b onto the column space of A. Let $p(b)$ be this projection. Then $AA^+b = Ax^+ = p(b)$. It follows that AA^+ is the projection matrix onto the column space of A.

Since x^+ is the *least norm* solution to $Ax = p(b)$, it must lie in the row space of A.

Consider any $y \in \mathcal{C}^n$. Let $b = Ay$ and $x^+ = A^+b = A^+Ay$. Since b is in the column space of A, $p(b) = b$. It follows that $Ax^+ = p(b) = Ay$, so $A(x^+ - y) = 0$. Thus $x^+ = A^+Ay$ is the orthogonal projection onto the row space of A.

(b) Observe that $A^+b = A^+p(b)$. Thus, for any b, $A^+AA^+b = A^+Ax^+ = A^+p(b) = A^+b$. Since this is true for all b, $A^+AA^+ = A^+$. Similarly, for any y, $AA^+Ay = Ax^+ = Ay$. Thus, $AA^+A = A$.

(c) Σ^+ is the diagonal matrix with entries

$$\Sigma^+_{ii} = \begin{cases} 1/\Sigma_{ii}, & \text{if } \Sigma_{ii} \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that $\Sigma\Sigma^+ = \Sigma^+\Sigma$ is diagonal (and hence Hermitian); $\Sigma\Sigma^+\Sigma = \Sigma$, and $\Sigma^+\Sigma\Sigma^+ = \Sigma^+$. Thus Σ^+ is the pseudoinverse.

(d) $A^+ = W\Sigma^+V^*$. To prove that this is the pseudoinverse, check each of the properties. Let $D := \Sigma\Sigma^+$, and observe that D is diagonal with $D_{ii} = 0$ if $\Sigma_{ii} = 0$, and $D_{ii} = 1$ otherwise. Then

- $AA^+ = V\Sigma W^*W\Sigma^+V^* = V\Sigma\Sigma^+V^* = VDV^*$, which is clearly Hermitian.
- Similarly, $A^+A = W\Sigma^+V^*V\Sigma W^* = WDW^*$, which is Hermitian.
- $AA^+A = V\Sigma W^*W\Sigma^+V^*V\Sigma W^* = VD\Sigma W^* = V\Sigma W^* = A$.
- $A^+AA^+ = W\Sigma^+V^*V\Sigma W^*V\Sigma^+V^* = WD\Sigma^+W^* = W\Sigma^+W^* = A^+$.