University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam

1 June 2009, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value unless stated otherwise..
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathbb{R}$ denotes the field of real numbers, and F denotes a field which may be either $\mathcal{C}$ or $\mathbb{R}$. $\mathcal{C}^n$ and $\mathbb{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively, written as column vectors. For $T \in \mathcal{L}(V)$, the *image* (sometimes called the *range*) of T is denoted $\mathrm{Im}(T)$. If A is a matrix over a field, then $\mathrm{rk}(A)$ is the *rank of* A. For $x \in \mathbb{R}^n$, $\|\cdot\|$ denotes the usual Euclidean norm. If A is an $m \times n$ matrix over F, T_A is the linear map defined by

$$T_A \colon F^n \to F^m \colon x \mapsto Ax.$$

  T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________

Total ________

1. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Define $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by

$$T\colon B \mapsto AB - BA.$$

   (i) (8 points) Fix an ordered basis $\mathcal{B}$ of $M_2(\mathbb{R})$ and compute the matrix $[T]_{\mathcal{B}}$ that represents T with respect to this basis.

   (ii) (8 points) Compute a basis for each of the eigenspaces of T.

   (iii) (4 points) Give the minimal and characteristic polynomials of T and the Jordan form for T.

2. Let $A \in M_6(\mathcal{C})$ be defined by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

   Find all of the eigenvalues, eigenvectors, and generalized eigenvectors of A. Construct the characteristic polynomial, the minimal polynomial, and the Jordan form of A.

3. Norms of Linear Operators

   (a) Let A be an $m \times n$ real matrix. Prove that there is a real constant M_A such that $\|A\mathbf{x}\| \leq M_A \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

   (b) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Prove that there is some positive constant $\|T\|$ for which

$$\|T(\mathbf{v})\| \leq \|T\| \cdot \|\mathbf{v}\|$$

   for all $\mathbf{v} \in \mathbb{R}^n$.

4. Spheres in Finite Dimensional Real Vector Spaces

   Let $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ be an ordered basis of the real vector space V with dimension n. For each $v \in V$ there are unique scalars $c_1, \ldots, c_n \in \mathbb{R}$ for which $v = \sum_{i=1}^n c_i v_i$. Write the *coordinate matrix* $[v]_{\mathcal{B}}$ *of* v *with respect to the ordered basis* $\mathcal{B}$ as

$$[v]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad \text{so that} \quad v = (v_1, \ldots, v_n) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \mathcal{B} \cdot [v]_{\mathcal{B}}.$$

For $\mathbf{c} = [c_1, \ldots, c_n]^T \in \mathbb{R}^n$, we employ the usual Euclidean norm:

$$\|\mathbf{c}\| = \sqrt{\sum_{i=1}^{n} c_i^2}.$$

For an arbitrary ordered basis $\mathcal{B}$ of V, we define the norm with respect to $\mathcal{B}$ as follows:

$$\|v\|_{\mathcal{B}} := \|[v]_{\mathcal{B}}\|.$$

Given the basis $\mathcal{B}$, a specific vector v_0 and a positive number r we can define the n-dimensional sphere with center v_0 and radius r (with respect to $\mathcal{B}$) by

$$S_{r,\mathcal{B}}(v_0) = \{w \in V : \|v_0 - w\|_{\mathcal{B}} \leq r\}.$$

Problem Let $r > 0$ and let $\mathcal{B}$, $\mathcal{B}'$ be any two ordered bases of V. Show that there is an $r' > 0$ such that

$$S_{r,\mathcal{B}}(\mathbf{0}) \subseteq S_{r',\mathcal{B}'}(\mathbf{0}).$$

5. Fredholm Alternative

   Let A be an $m \times n$ real matrix and $b \in \mathbb{R}^m$. Show that exactly one of the following systems has a solution:

   i) $Ax = b$
   ii) $A^T y = 0, \quad y^T b \neq 0.$

   Note: Our notation is $y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, so $y^T = [y_1, \ldots, y_m]$.

6. Upper-triangularization

   (a) (12 points) For each of the following, if it is true, merely say so; if it is false, give a counterexample.

   (i) If V is a finite-dimensional vector space over $\mathbb{R}$ and $T \in \mathcal{L}(V)$, then V has a basis $\mathcal{B}$ with respect to which $[T]_{\mathcal{B}}$ is upper triangular.

   (ii) If V is a finite-dimensional vector spacee over $\mathcal{C}$ and $T \in \mathcal{L}(V)$, then V has a basis $\mathcal{B}$ with respect to which $[T]_{\mathcal{B}}$ is upper triangular.

   (iii) If V is a finite-dimensional vector space over $\mathcal{C}$ and $S, T \in \mathcal{L}(V)$, then V has a basis $\mathcal{B}$ for which both $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ are upper triangular.

   (b) (8 points) Show that a normal, upper triangular matrix must be diagonal.

7. Tournament Matrices

   The matrices of this problem are all $n \times n$ with real entries.

(a) Show that if the matrix A is skew-symmetric then $I + A$ is nonsingular.

(b) Show that for arbitrary matrices A and B, $\mathrm{rk}(A + B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$.

(c) If A is arbitrary and J is the matrix of all 1's, then show that

$$\mathrm{rk}(A - J) \geq \mathrm{rk}(A) - \mathrm{rk}(J).$$

(d) If M is a $(0, 1)$-matrix with zeros on the main diagonal and with $M_{ij} = 0$ if and only if $M_{ji} = 1$, show that $\mathrm{rk}(M) \geq n - 1$. (Such a matrix is called a *tournament* matrix.)

8. Given an $m \times n$ matrix A, the *pseudoinverse of* A, denoted A^+, can be defined as the matrix such that for all $b \in \mathcal{C}^m$, $x^+ := A^+b$ is the least squares solution to the equation $Ax = b$ that has the smallest norm.

(a) Using the above definition, explain why AA^+ and A^+A must be projection matrices (and are therefore Hermitian). Onto what fundamental subspaces do these matrices project?

(b) Prove that $AA^+A = A$ and $A^+AA^+ = A^+$. (Note: these two properties, together with the Hermitian properties in part (a) uniquely determine the pseudoinverse).

(c) If Σ is a real diagonal matrix, what is Σ^+?

(d) Give an explicit formula for A^+ in terms of the singular value decomposition $A = V\Sigma W^*$. Justify your answer.