University of Colorado at Denver — Mathematics Department

Applied Linear Algebra Preliminary Exam With Solutions

16 January 2009, 10:00 am – 2:00 pm

Name: ____

The proctor will let you read the following conditions before the exam begins, and you will have time for questions. Once the exam begins, you will have 4 hours to do your best. This is a closed book exam. Please put your name on each sheet of paper that you turn in.

Exam conditions:

- Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points; parts of problems have equal value.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write legibly using a dark pencil or pen.
- Notation: $\mathcal{C}$ denotes the field of complex numbers , $\mathcal{R}$ denotes the field of real numbers, and F denotes a field which may be either $\mathcal{C}$ or $\mathcal{R}$. $\mathcal{C}^n$ and $\mathcal{R}^n$ denote the vector spaces of n-tuples of complex and real scalars, respectively. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. v^T and A^T denote vector and matrix transposes, respectively.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

On this exam V is a finite dimensional vector space over the field F, where either $F = \mathcal{C}$, the field of complex numbers, or $F = \mathcal{R}$, the field of real numbers. Also, F^n denotes the vector space of column vectors with n entries from F, as usual. For $T \in \mathcal{L}(V)$, the *image* (sometimes called the *range*) of T is denoted $\mathrm{Im}(T)$.

1. Suppose that $P \in \mathcal{L}(V)$ (the vector space of linear maps from V to itself) and that $P^2 = P$.

   (a) (6 points) Determine all possible eigenvalues of P.

   (b) (10 points) Prove that $V = \mathrm{null}(P) \oplus \mathrm{Im}(P)$.

   (c) (4 points) Is it necessary that all possible eigenvalues found in part (a) actually must occur? Prove that your answer is correct.

   Solution: $P^2 - P = \mathbf{0}$ implies that the minimal polynomial $p(x)$ of P divides $x^2 - x = x(x-1)$. Hence $p(x) = x$, or $(x-1)$, or $x(x-1)$. So in general the eigenvalues are each equal to either 0 or 1. But $p(x) = x$ if and only if $P = 0$, in which case $V = \mathrm{null}(P)$ and $\{\mathbf{0}\} = \mathrm{Im}(P)$. And $p(x) = x - 1$ if and only if $P = I$. In this case $V = \mathrm{Im}(P)$ and $\mathrm{null}(P) = \{\mathbf{0}\}$. In these two cases the condition in part (b) clearly holds, and we see that part (c) is also answered.

   Finally, suppose $p(x) = x(x-1)$, so that both 0 and 1 are eigenvalues of P. If $v \in \mathrm{null}(P) \cap \mathrm{Im}(P)$, then $P(v) = 0$ on the one hand, and on the other hand there is some $w \in V$ for which $v = P(w) = P^2(w) = P(v) = \mathbf{0}$. Hence $\mathrm{null}(P) \cap \mathrm{Im}(P) = \{\mathbf{0}\}$. But also for any $v \in V$ we have $v = (v - P(v)) + P(v)$, where $P(v - P(v)) = p(v) - P(v) = \mathbf{0}$. So $v - P(v) \in \mathrm{null}(P)$ and clearly $P(v) \in \mathrm{Im}(P)$. Hence $V = \mathrm{null}(P) \oplus \mathrm{Im}(P)$. This finishes part (b).

2. Define $T \in \mathcal{L}(F^n)$ by $T\colon (w_1, w_2, w_3, w_4)^T \mapsto (0, w_2 + w_4, w_3, w_4)^T$.

   (a) (8 points) Determine the minimal polynomial of T.

   (b) (6 points) Determine the characteristic polynomial of T.

   (c) (6 points) Determine the Jordan form of T.

   Solution: Let $p(x)$ be the minimal polynomial of T. It is easy to see that $T(1,0,0,0) = \mathbf{0}$, so 0 is an eigenvalue of T and hence x is a divisor of $p(x)$. Also, $T(0,1,0,0) = (0,1,0,0)$, so 1 is an eigenvalue of T and $x-1$ divides $p(x)$. Since $T^2(x_1, x_2, x_3, x_4) = (0, x_2 + 2x_4, x_3, x_4)$, it is clear that $\mathrm{null}(T) = \mathrm{null}(T^2) = \{a, 0, 0, 0) : a \in F\}$, hence the dimension of the space of generalized eigenvectors of T associated with 0 is 1. This says that the multiplicity of 0 as a root of the characteristic polynomial $f(x)$ of T is 1. So we check for eigenvalue 1. $(T - I)(x_1, x_2, x_3, x_4) = (-x_1, x_4, 0, 0)$. Repeating this we see $(T - I)^2(x_1, x_2, x_3, x_4) = (x_1, 0, 0, 0)$, which is in the null space of T. Hence $T(T - I)^2 = \mathbf{0}$. Since $T(T - I)(x_1, x_2, x_3, x_4) = (0, x_4, 0, 0)$, clearly $T(T - I)$ is not the zero operator, hence $p(x) = x(x-1)^2$. This finishes part (a).

   Part (b): Since the dimension of the space of generalized eigenvectors belonging to 0 is 1, it must be that the dimension of the space of generalized eigenvectors belonging to 1 is 3. Hence the characteristic polynomial of T must be $f(x) = x(x-1)^3$.

Part (c) Since the minimal polynomial of T is $x(x-1)^2$ and the characteristic polynomial is $x(x-1)^3$, the only possibility (up to the order of the diagonal blocks) for the Jordan form of T is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Let T be a normal operator on a complex inner product space V of dimension n.

   (a) (10 points) If $T(v) = \lambda v$ with $\mathbf{0} \neq v \in V$, show that v is an eigenvector of the adjoint T^* with associated eigenvalue $\overline{\lambda}$.

   (b) (10 points) Show that T^* is a polynomial in T.

   Solution to part (a):

$$\begin{aligned} T(v) = \lambda v \quad &\Leftrightarrow \quad 0 = \|(T - \lambda I)(v)\|^2 \\ = \langle (T-\lambda I)v, (T-\lambda I)v\rangle \quad &= \quad \langle v, (T^* - \overline{\lambda} I)(T - \lambda I)v\rangle \\ = \langle v, (T-\lambda I)(T^* - \overline{\lambda} I)v\rangle \quad &= \quad \|(T^* - \overline{\lambda} I)v\|^2 \\ &\Leftrightarrow \quad T^*(v) = \overline{\lambda} v. \end{aligned}$$

   Solution to part (b): Since T is a normal operator on a complex vector space V, there is an orthonormal basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V consisting of eigenvectors of T. Suppose that $T(v_i) = \lambda_i v_i$ for $1 \leq i \leq n$. So by part (a) we know that $T^*(v_i) = \overline{\lambda}_i v_i$, for $1 \leq i \leq n$. WLOG we may assume that the eigenvalues have been ordered so that $\lambda_1, \lambda_2, \dots, \lambda_r$ are the distinct eigenvalues of T. Using Lagrange interpolation (or any method have at hand) construct a polynomial $f(x) \in \mathcal{C}[x]$ (having degree at most $r-1$, if desired), such that $f(\lambda_i) = \overline{\lambda}_i$, for $1 \leq i \leq r$. Then $f(T)(v_j) = f(\lambda_j)(v_j) = \overline{\lambda}_j(v_j) = T^*(v_j)$, $1 \leq j \leq n$, so that $f(T)$ and T^* have the same effect on each member of the basis $\mathcal{B}$. This implies that $f(T) = T^*$.

4. Let A and B be $n \times n$ Hermitian matrices over $\mathcal{C}$.

   (a) (10 points) If A is positive definite, show that there exists an invertible matrix P such that $P^*AP = I$ and P^*BP is diagonal.

   (b) (10 points) If A is positive definite and B is positive semidefinite, show that

$$\det(A + B) \geq \det(A).$$

   Solution:

   (a) Since A is positive definite, there exists an invertible matrix T such that $A = T^*T$. $(T^{-1})^*B(T^{-1})$ is Hermitian, so is diagonalizable. That is, there exists a unitary matrix U and a diagonal matrix D such that $U^*(T^{-1})^*B(T^{-1})U = D$. Let $P = T^{-1}U$. Then $P^*BP = D$, and

$$P^*AP = U^*(T^{-1})^*(T^*T)T^{-1}U = U^*U = I.$$

(b) Let P and D be as defined above. Then $A = (P^*)^{-1}P^{-1}$ and $B = (P^*)^{-1}DP^{-1}$. Since B is positive semidefinite, then the diagonal entries in D are nonnegative. Thus

$$\begin{aligned}\det(A+B) &= \det\left((P^*)^{-1}(I+D)P^{-1}\right) = \det\left((P^*)^{-1}P^{-1}\right)\det(I+D) \\ &= \det A \det(I+D) \geq \det A.\end{aligned}$$

5. Let $\|\cdot\|_\infty : \mathcal{C}^n \to \mathcal{R}$ be defined by

$$\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n}|x_i|.$$

(a) (8 points) Prove that $\|\cdot\|_\infty$ is a norm.

(b) (12 points) A norm $\|\cdot\|$ is said to be derived from an inner product if there is an inner product $\langle\cdot,\cdot\rangle$ such that $\|\mathbf{x}\| = \langle\mathbf{x},\mathbf{x}\rangle^{1/2}$ for all $\mathbf{x}\in\mathcal{C}^n$. Show that $\|\cdot\|_\infty$ cannot be derived from an inner product.

Solution:

(a) We verify the properties of norms:

i. $\|x\|_\infty = \max_{1\leq i\leq n}|x_i| \geq 0$, for all $x\in\mathbb{C}^n$.

ii. $\|x\|_\infty = 0 \iff \max_{1\leq i\leq n}|x_i| = 0 \iff x = 0$.

iii. For any $c\in\mathbb{C}$ and $x\in\mathbb{C}^n$, $\|cx\|_\infty = \max_{1\leq i\leq n}|cx_i| = |c|\max_{1\leq i\leq n}|x_i| = |c|\,\|x\|_\infty$.

iv. For all $x,y\in\mathbb{C}^n$, $\|x+y\|_\infty = \max_{1\leq i\leq n}|x_i+y_i| \leq \max_{1\leq i\leq n}|x_i|+|y_i| \leq \max_{1\leq i\leq n}|x_i| + \max_{1\leq i\leq n}|y_i| = \|x\|_\infty + \|y\|_\infty$.

(b) Assume there exists an inner product $\langle\cdot,\cdot\rangle$ such that $\|x\|_\infty = \langle x,x\rangle^{1/2}$ for all $x\in\mathbb{C}^n$. Then for any $x,y\in\mathbb{C}^n$, we have

$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 2\langle x,x\rangle + 2\langle y,y\rangle = 2\|x\|_\infty^2 + 2\|y\|_\infty^2.$$

But, choosing $x = (1,0,\ldots,0)^T$ and $y = (0,1,0,\ldots,0)^T$, this yields the following contradiction:

$$2 = \|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2 = 2+2 = 4.$$

(One of our theorems said that a norm is derived from an inner product if and only if it satisfies the parallelogram equality, so this type of proof should naturally come to mind.)

6. Suppose that $F = \mathcal{C}$ and that S, $T\in\mathcal{L}(V)$ satisfy $ST = TS$. Prove each of the following:

(a) (4 points) If λ is an eigenvalue of S, then the eigenspace

$$V_\lambda = \{\mathbf{x}\in V | S\mathbf{x} = \lambda\mathbf{x}\}$$

is invariant under T.

(b) (4 points) S and T have at least one common eigenvector (not necessarily belonging to the same eigenvalue).

(c) (12 points) There is a basis $\mathcal{B}$ of V such that the matrix representations of S and T are both upper triangular.

Solution:

(a) If $x \in V_\lambda$, then $Sx = \lambda x$. Thus,

$$S(Tx) = TSx = T(\lambda x) = \lambda Tx,$$

so $Tx \in V_\lambda$.

(b) Let $T_{|V_\lambda}$ denote the restriction of T to the subspace V_λ. $T_{|V_\lambda}$ has at least one eigenvector $v \in V_\lambda$, with eigenvalue μ. It follows that $Tv = T_{|V_\lambda}v = \mu v$, so v is an eigenvector of V. And since $v \in V_\lambda$, it is also an eigenvector of S.

(c) The matrix of a linear transformation with respect to a basis $\{v_1, \ldots, v_n\}$ is upper triangular if and only if $\text{span}(v_1, \ldots, v_k)$ is invariant for each $k = 1, \ldots, n$. Using part (b) above, we shall construct a basis $\{v_1, \ldots, v_n\}$ for V such that $\text{span}(v_1, \ldots, v_k)$ is invariant under both S and T for each k.

We proceed by induction on n, the dimension of V, with the result being clearly true if $n = 1$. So suppose that $n > 1$ with the desired result holding for all operators on spaces of positive dimension less than n. By part (b) there is a vector $v_1 \in V$ such that $Tv_1 = \lambda_1 v_1$ and $Sv_1 = \mu_1 v_1$ for some scalars λ_1 and μ_1. Let W be the subspace spanned by v_1. Then the dimension of the quotient space V/W is $n-1$, and the operators $\overline{T}$ and $\overline{S}$ induced on V/W commute, so by our induction hypothesis there is a basis $\mathcal{B}_1 = (v_2 + W, v_3 + W, \ldots, v_n + W)$ of V/W with respect to which both $\overline{T}$ and $\overline{S}$ have upper triangular matrices. It follows that $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ is a basis of V with respect to which both T and S have upper triangular matrices.

7. Let $F = \mathcal{C}$ and suppose that $T \in \mathcal{L}(V)$.

(a) (10 points) Prove that the dimension of $\text{Im}(T)$ equals the number of nonzero singular values of T.

(b) (10 points) Suppose that $T \in \mathcal{L}(V)$ is positive semidefinite. Prove that T is invertible if and only if $\langle T(\mathbf{x}), \mathbf{x}\rangle > 0$ for every $\mathbf{x} \in V$ with $\mathbf{x} \neq \mathbf{0}$.

Solution:

Let $T \in \mathcal{L}(V)$. Since T^*T is self-adjoint, there is an orthonormal basis $(v_1, \ldots, v_n)$ of V whose members are eigenvectors of T^*T, say $T^*Tv_j = \lambda_j v_j$, for $1 \le j \le n$. Note $\|Tv_j\|^2 = \langle Tv_j, Tv_j\rangle = \langle T^*Tv_j, v_j\rangle = \lambda_j \|v_j\|^2$, so in particluar $\lambda_j \ge 0$.

Then T^*T has real, nonnegative eigenvalues, so we may suppose they are $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$. Put $s_i = \sqrt{\lambda_i}$, $1 \le i \le n$, so that $s_1 \ge s_2 \ge \cdots \ge s_r > 0$ are the nonzero singular values of T and in general $s_i = \|Tv_i\|$, for all $i = 1, 2, \ldots, n$. It follows that $(v_{r+1}, v_{r+2}, \ldots, v_n)$ is a basis of the null space of T and $(Tv_1, Tv_2, \ldots, Tv_r)$ is a basis for the Image of T. Clearly r is the number of nonzero singular values and also the dimension of the range of T, finishing part (a).

(b) Suppose that $T \in \mathcal{L}(V)$ is positive semidefinite, i.e., T is self-adjoint and $\langle T(v), v\rangle \ge 0$ for all $v \in V$. Since T is self-adjoint we know there is an operator S for which $T = S^*S$. So $\langle T(v), v\rangle = \langle S^*S(v), v\rangle = \langle S(v), S(v)\rangle = 0$ if and only if $S(v) = \mathbf{0}$. So T

is invertible if and only if S is invertible iff $S(v) \neq \mathbf{0}$ whenever $v \neq \mathbf{0}$ iff $\langle T(v), v\rangle > 0$ whenever $v \neq \mathbf{0}$.

8. Let N be a real $n \times n$ matrix of rank $n-m$ and nullity m. Let L be an $m \times n$ matrix whose rows form a basis of the left null space of N, and let R be an $n \times m$ matrix whose columns form a basis of the right null space of N. Put $Z = L^T R^T$. Finally, put $M = N + Z$.

   (a) (2 points) For $\mathbf{x} \in \mathcal{R}^n$, show that $N^T\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = L^T\mathbf{y}$ for some $\mathbf{y} \in \mathcal{R}^m$.

   (b) (2 points) For $\mathbf{x} \in \mathcal{R}^n$, show that $N\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = R\mathbf{y}$ for some $\mathbf{y} \in \mathcal{R}^m$.

   (c) (4 points) Show that Z is an $n \times n$ matrix with rank m for which $N^T Z = \mathbf{0}$, $NZ^T = \mathbf{0}$ and $MM^T = NN^T + ZZ^T$.

   (d) (12 points) Show that the eigenvalues of MM^T are precisely the positive eigenvalues of NN^T and the positive eigenvalues of ZZ^T, and conclude that MM^T is nonsingular.

Solution:

(a) $N^T X = 0$ iff $X^T N = 0$ iff X^T is in the row space of L, i.e., iff $X^T = Y^T L$ for some $Y \in \mathcal{R}^m$, iff $X = L^T Y$ for some $Y \in \mathcal{R}^m$.

(b) $NX = 0$ iff X is in the column space of R, i.e., iff $X = RY$ for some $Y \in \mathcal{R}^m$.

(c) $N^T Z = N^T L^T R^T = 0$ by part (a). Similarly, $NZ^T = NRL = 0$ by part (b). The columns of L^T are independent and m in number, so $Zv = 0$ iff $L^T(R^T v) = 0$ iff $R^T v = 0$. Since R^T is $m \times n$ with rank m and right nullity $n-m$, $Z = L^T R^T$ must have nullity $n-m$, and hence rank m. It now is easy to compute $MM^T = (N+Z)(N^T + Z^T) = NN^T + NZ^T + ZN^T + ZZ^T = NN^T + ZZ^T$.

(d) NN^T and ZZ^T are real, symmetric commuting matrices (both products are 0), so there must be an orthogonal basis $\mathcal{B} = (v_1, v_2, \ldots, v_n)$ of $\mathcal{R}^n$ consisting of eigenvectors of both NN^T and ZZ^T. We know that all the eigenvalues of NN^T and ZZ^T are real and nonnegative. Suppose that v_i is a member of $\mathcal{B}$ for which $NN^T v_i = \lambda_i v_i \neq \mathbf{0}$. v_i must be orthogonal to all the vectors in the right null space of NN^T, i.e., v_i orthogonal to the right null space of N^T . This says $v_i^T Z = 0$, which implies $ZZ^T v_i = \mathbf{0}$. Hence each v_i not in the null space of NN^T must be in the null space of ZZ^T. N and Z play symmetric roles, so a similar argument shows that each v_j not in the right null space of ZZ^T must be in the null space of NN^T. Hence we may assume that the members of $\mathcal{B}$ are ordered so that $v_1, \ldots, v_{n-m}$ are not in the null space of NN^T and are in the null space of ZZ^T. Similarly, $v_{n-m+1}, \ldots, v_n$ are in the null space of NN^T and not in the null space of ZZ^T. It follows immediately that $v_1, \ldots, v_{n-m}$ are eigenvectors of MM^T belonging to the positive eigenvalues of NN^T and $v_{n-m+1}, \ldots, v_n$ are eigenvectors of MM^T belonging to the positive eigenvalues of ZZ^T. Finally, since all the eigenvalues of MM^T are positive (i.e., none of them is zero), MM^T is nonsingular.