

University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 22, 2016

Name: _____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. All solutions will be graded and your final grade will be based on your six best problems. Your final score will count out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in “essay-style” using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce an independent proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

1.		5.	
2.		6.	
3.		7.	
4.		8.	
		Total	

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Varis Carey, Stephen Hartke, and Julien Langou (Chair).

Problem 1

Let V be an inner product space over \mathbb{C} , with inner product $\langle u, v \rangle$.

- (a) Prove that any finite set S of nonzero, pairwise orthogonal vectors is linearly independent.
- (b) If $T : V \rightarrow V$ is a linear operator satisfying $\langle T(u), v \rangle = \langle u, T(v) \rangle$ for all $u, v \in V$, prove that all eigenvalues of T are real.
- (c) If $T : V \rightarrow V$ is a linear operator satisfying $\langle T(u), v \rangle = \langle u, T(v) \rangle$ for all $u, v \in V$, prove that the eigenvectors of T associated with distinct eigenvalues λ and μ are orthogonal.

Problem 2

- (a) Let A be a 2-by-2 real matrix of the form $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ where $\lambda \in \mathbb{R}$. Prove that A has a square root: that is, there exists a matrix B such that $B^2 = A$.
- (b) Prove that a real symmetric matrix having the property that every negative eigenvalue occurs with even multiplicity has a square root.

Problem 3

Let A and B be two complex square matrices, and suppose that A and B have the same eigenvectors. Show that if the minimal polynomial of A is $(x + 1)^2$ and the characteristic polynomial of B is x^5 , then $B^3 = 0$.

Problem 4

Let A be an m -by- n complex matrix. Let $B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$. Prove that $\|B\|_2 = \|A\|_2$.

Problem 5

Given the 2-by-2 real matrix $A = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$, determine the set of all real a, b such that

- (a) A is orthogonal.
- (b) A is symmetric positive definite.
- (c) A is nilpotent.
- (d) A is unitarily diagonalizable.

Problem 6

Note: Question (a) and (b) are not related.

- (a) We consider $\mathcal{C}([0, 1])$, the space of continuous function on $[0, 1]$. We dot $\mathcal{C}([0, 1])$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

We consider the subspace \mathcal{P}_1 of all polynomials of degree 1 or less on the unit interval $0 \leq x \leq 1$.

Find the least squares approximation to the function $f(x) = x^3$ by a polynomial $p \in \mathcal{P}_1$ on the interval $[0, 1]$, i.e., find $p \in \mathcal{P}_1$ that minimizes $\|p - f\|_2$.

- (b) We consider the vector space \mathcal{P}_2 of all polynomials of degree 2 or less on the unit interval $0 \leq x \leq 1$. We consider the set of functions

$$S = \{p \in \mathcal{P}_2 : \int_0^1 p(x)dx = \int_0^1 p'(x)dx\}.$$

Show that this is a linear subspace of \mathcal{P}_2 , determine its dimension and find a basis for \mathcal{S} .

Problem 7

Let E , F , and G be vector spaces. Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$. Prove that:

$$\text{Range}(g \circ f) = \text{Range}(g) \iff \text{Null}(g) + \text{Range}(f) = F$$

Problem 8

Let A and B be $n \times n$ complex matrices such that $AB = BA$. Show that if A has n distinct eigenvalues, then A , B , and AB are all diagonalizable.