University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 22, 2016

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. All solutions will be graded and your final grade will be based on your six best problems. Your final score will count out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce an independent proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

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Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

Let V be an inner product space over $\mathbb{C}$, with inner product $\langle u, v\rangle$.

(a) Prove that any finite set S of nonzero, pairwise orthogonal vectors is linearly independent.

(b) If $T : V \to V$ is a linear operator satisfying $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in V$, prove that all eigenvalues of T are real.

(c) If $T : V \to V$ is a linear operator satisfying $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in V$, prove that the eigenvectors of T associated with distinct eigenvalues λ and μ are orthogonal.

Solution:

(Note in the solution we write $T(u)$ as Tu. This is standard notation.)

(a) Let $S = (v_1, \ldots, v_n)$ be a finite set of n nonzero, pairwise orthogonal vectors. We want to prove that $(v_1, \ldots, v_n)$ is linearly independent. Let $\alpha_1, \ldots, \alpha_n$ be n complex numbers such that

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = 0.$$

We want to prove that $\alpha_1 = \ldots = \alpha_n = 0$. Applying $\langle v_1, \cdot\rangle$ to the previous equality, we get

$$0 = \langle v_1, \alpha_1 v_1 + \ldots + \alpha_n v_n\rangle = \alpha_1\langle v_1, v_1\rangle + \ldots \alpha_n\langle v_1, v_n\rangle = \alpha_1\langle v_1, v_1\rangle.$$

(First equality is what we start with. Second is first slot linearity of $\langle\cdot,\cdot\rangle$. Third is pairwise orthogonality.) So we get $\alpha_1\langle v_1, v_1\rangle = 0$. But v_1 is nonzero, so $\langle v_1, v_1\rangle$ is not zero, and so this previous equality implies $\alpha_1 = 0$. With a similar method repeated $n-1$ times, we get $\alpha_2 = \ldots = \alpha_n = 0$.

(b) Let $T : V \to V$ be a self-adjoint linear operator. (So that we have $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in V$.) Let λ be an eigenvalue of T. We want to prove that λ is real. Let v be an associated (nonzero) eigenvector of λ. We have

$$Tv = \lambda v.$$

Applying $\langle v, \cdot\rangle$ to the previous equality, we get

$$\langle v, Tv\rangle = \langle v, \lambda v\rangle = \lambda\langle v, v\rangle.$$

Applying $\langle\cdot, v\rangle$ to the same equality, we get

$$\langle Tv, v\rangle = \langle \lambda v, v\rangle = \bar{\lambda}\langle v, v\rangle.$$

Since $\langle v, Tv\rangle = \langle Tv, v\rangle$, we get

$$\lambda\langle v, v\rangle = \bar{\lambda}\langle v, v\rangle.$$

Since v is nonzero, so $\langle v, v\rangle$ is not zero, and so this previous equality implies

$$\lambda = \bar{\lambda}.$$

So λ is real.

(c) Let $T : V \to V$ be a self-adjoint linear operator. (So that we have $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in V$.) Let (u, λ) and (v, μ) be two eigencouples of T where λ and μ are distinct. We want to prove that $\langle u, v\rangle = 0$. We have that $Tu = \lambda u$, applying $\langle \cdot, v\rangle$, we get

$$\langle Tu, v\rangle = \langle \lambda u, v\rangle = \bar{\lambda}\langle u, v\rangle = \lambda\langle u, v\rangle.$$

(The last equality comes from part (b): λ is real.) We also have that $Tv = \mu v$, applying $\langle u, \cdot\rangle$, we get

$$\langle u, Tv\rangle = \langle u, \mu v\rangle = \mu\langle u, v\rangle.$$

But $\langle u, Tv\rangle = \langle Tu, v\rangle$, so we get that

$$\lambda\langle u, v\rangle = \mu\langle u, v\rangle,$$

in other words,

$$(\lambda - \mu)\langle u, v\rangle = 0.$$

We have that λ and μ are distinct, so $(\lambda - \mu) \neq 0$, and so we must have

$$\langle u, v\rangle = 0.$$

Problem 2

(a) Let A be a 2-by-2 real matrix of the form $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ where $\lambda \in \mathbb{R}$. Prove that A has a square root: that is, there exists a matrix B such that $B^2 = A$.

(b) Prove that a real symmetric matrix having the property that every negative eigenvalue occurs with even multiplicity has a square root.

Solution:

(a) Let A be a 2-by-2 real matrix of the form $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ where $\lambda \in \mathbb{R}$.

Method 1: Take $B = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$. Observe that $B^2 = A$.

Method 2: (Much longer but might be more intuitive.) Two cases:

(a) Either λ is nonnegative in which case we have that a square root of A is $B = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{1/2} \end{pmatrix}$.

(b) Or λ is negative in which case we have that a square root of A is $B = \begin{pmatrix} 0 & -(-\lambda)^{1/2} \\ (-\lambda)^{1/2} & 0 \end{pmatrix}$.

For both cases, we exhibited a square root of A.

We note that, for the negative case, we were inspired by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This can be understood by saying that two 90-degree rotation is same as a 180-degree rotation.

(b) Let A be a real symmetric matrix having the property that every negative eigenvalue occurs with even multiplicity.

(We note that the question does not specify whether the multiplicity is algebraic or geometric. Since the matrix is symmetric, we remind that the geometric multiplicity and the algebraic multiplicity are the same.)

Since A is symmetric, we can use the spectral theorem to diagonalize A in an orthonormal basis. So we have that there exist real matrices V and D such that $A = VDV^T$, D is diagonal with the eigenvalues of A on its diagonal ordered from the smallest to largest, V is orthogonal matrix (i.e. $VV^T = V^TV = I$).

Now we pay special attention to matrix D. We explain why D has a square root, $D^{1/2}$, below. We call λ_i^-, ..., λ_1^-, the i negative eigenvalues of A ordered from smallest to largest. We call λ_1^+, ..., λ_j^+, the j nonnegative eigenvalues of A ordered from smallest to largest. The matrix D therefore is made of scalar diagonal blocks of the form λI. It looks like:

$$D = \begin{pmatrix} D_{\lambda_i^-} & & & & & \\ & \ddots & & & & \\ & & D_{\lambda_1^-} & & & \\ & & & D_{\lambda_1^+} & & \\ & & & & \ddots & \\ & & & & & D_{\lambda_j^+} \end{pmatrix}.$$

By part (a), we see that each of the D_λ block has a square root. Either λ is nonnegative and then simply $B_\lambda = \lambda^{1/2} I$. Or λ is negative, in which case the dimension of D_λ is even, (by assumption of the problem) and so we can split D_λ in 2-by-2 identical scalar blocks of the form $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, which each has a square root, (see part (a),) for example $\begin{pmatrix} 0 & -(-\lambda)^{1/2} \\ (-\lambda)^{1/2} & 0 \end{pmatrix}$. So we see that each of the D_λ block has a square root, B_λ, whether λ is nonnegative or negative. And so we set

$$B_D = \begin{pmatrix} B_{\lambda_i^-} & & & & & \\ & \ddots & & & & \\ & & B_{\lambda_1^-} & & & \\ & & & B_{\lambda_1^+} & & \\ & & & & \ddots & \\ & & & & & B_{\lambda_j^+} \end{pmatrix}.$$

And we have

$$D = B_D B_D.$$

This explains why D has a square root, B_D.

So now we call $B_A = V B_D V^T$ and we see that

$$A = VDV^T = VB_DB_DV^T = VB_D(V^TV)B_DV^T = (VB_DV^T)(VB_DV^T) = B_AB_A.$$

So that B_A is a square root of A.

Problem 3

Let A and B be two complex square matrices, and suppose that A and B have the same eigenvectors. Show that if the minimal polynomial of A is $(x+1)^2$ and the characteristic polynomial of B is x^5, then $B^3 = 0$.

Solution: Since A and B have the same eigenvectors, the matrices A and B have the same dimension.
(Note: since we are working with complex square matrices, matrices A and B have at least one eigenvector, and so the statement "A and B have the same eigenvectors" cannot be vacuously true.)
Since the characteristic polynomial of B is of degree 5, matrices B is 5-by-5; consequently, so is A.
Since the minimal polynomial of A is $(x+1)^2$, we see that the Jordan form of A is

(a) either

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

so 2 Jordan blocks of size 2 and 1 Jordan block of size 1, in this case $\dim(\text{Null}(A+I)) = 3$;

(b) or

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

so 1 Jordan block of size 2 and 3 Jordan blocks of size 1, in this case $\dim(\text{Null}(A+I)) = 4$.

So either $\dim(\text{Null}(A+I))$ is 3 or 4.
Since the characteristic polynomial of B is x^5, 0 is the only eigenvalue of B.
Since A and B have the same eigenvectors, and since B has only one eigenvalue, 0, and since $\dim(\text{Null}(A+I))$ is 3 or 4, then $\dim(\text{Null}(B))$ is either 3 or 4 as well. So B has either 3 or 4 Jordan blocks.
This means that the possible Jordan block structures for B are

(a) (3 Jordan blocks, $\dim(\text{Null}(B)) = 3$), B has 2 Jordan blocks of size 2 and 1 Jordan block of size 1,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In that case, the minimal polynomial of B is x^2. And so $B^2 = 0$. And so indeed $B^3 = 0$.

(b) (3 Jordan blocks, $\dim(\text{Null}(B)) = 3$), B has 1 Jordan block of size 3 and 2 Jordan blocks of size 1,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In that case, the minimal polynomial of B is x^3. And so $B^3 = 0$.

(c) (4 Jordan blocks, $\dim(\text{Null}(B)) = 4$), B has 1 Jordan block of size 2 and 3 Jordan blocks of size 1,

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In that case, the minimal polynomial of B is x^2. And so $B^2 = 0$. And so indeed $B^3 = 0$.

In all three cases, we see that $B^3 = 0$.

Problem 4

Let A be an m-by-n complex matrix. Let $B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$. Prove that $\|B\|_2 = \|A\|_2$.

Solution: By definition of the 2-norm, we have that

$$\|A\|_2 = \max_{x \in \mathbb{R}^n \text{s.t.} \|x\|_2 = 1} \|Ax\|_2,$$

$$\|A^*\|_2 = \max_{y \in \mathbb{R}^m \text{s.t.} \|y\|_2 = 1} \|A^* y\|_2,$$

$$\|B\|_2 = \max_{z \in \mathbb{R}^{(m+n)} \text{s.t.} \|z\|_2 = 1} \|Bz\|_2.$$

We also know that

$$\|A\|_2 = \|A^*\|_2.$$

Let x in $\mathbb{R}^n$ such that $\|x\|_2 = 1$ and $\|Ax\|_2 = \|A\|_2$. Consider the vector z in $\mathbb{R}^{m+n}$

$$z = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Now we have that

$$\|z\|_2 = \| \begin{pmatrix} x \\ 0 \end{pmatrix} \|_2 = \|x\|_2 = 1.$$

$$\text{and } \|Bz\|_2 = \| \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} \|_2 = \| \begin{pmatrix} 0 \\ Ax \end{pmatrix} \|_2 = \|Ax\|_2 = \|A\|_2.$$

Since $\|B\|_2 = \max_{z \in \mathbb{R}^{(m+n)} \text{s.t.} \|z\|_2 = 1} \|Bz\|_2$, and since we have found a z such that $\|z\|_2 = 1$ and $\|Bz\|_2 = \|A\|_2$, this proves that

$$\|A\|_2 \leq \|B\|_2.$$

Let z in $\mathbb{R}^{(m+n)}$ such that $\|z\|_2 = 1$. Let x in $\mathbb{R}^n$ and y in $\mathbb{R}^m$ such that

$$z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

We have $\sqrt{\|x\|_2^2 + \|y\|_2^2} = 1$.
We now look at $\|Bz\|_2$. We have that

$$\|Bz\|_2 = \| \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \|_2 = \| \begin{pmatrix} A^* y \\ Ax \end{pmatrix} \|_2 = \sqrt{\|A^* y\|_2^2 + \|Ax\|_2^2}$$

But, on the one hand, $\|Ax\|_2 \leq \|A\|_2 \|x\|_2$. And, on the other hand, $\|A^* y\|_2 \leq \|A^*\|_2 \|y\|_2$ and, since $\|A\|_2 = \|A^*\|_2$, we get $\|A^* y\|_2 \leq \|A\|_2 \|y\|_2$. So

$$\|Bz\|_2 \leq \|A\|_2 \sqrt{\|x\|_2^2 + \|y\|_2^2},$$

and, since $\sqrt{\|x\|_2^2 + \|y\|_2^2} = 1$, we find

$$\|Bz\|_2 \leq \|A\|_2.$$

We proved that, for all z such that $\|z\|_2 = 1$, we have $\|Bz\|_2 \leq \|A\|_2$, so, since $\|B\|_2 = \max_{z\in\mathbb{R}^{(m+n)}\text{s.t.}\|z\|_2=1} \|Bz\|_2$, we get that

$$\|B\|_2 \leq \|A\|_2.$$

We can now conclude

$$\|B\|_2 = \|A\|_2.$$

<u>Note:</u> Another way to go about this problem is to know the relation between the SVD of A and the eigenvalues of B. For example, in the case where $m \geq n$, A has n singular values. We denote these singular values are $(\sigma_1, \dots, \sigma_n)$. Then B has $m+n$ eigenvalues and they are $(\sigma_1, \dots, \sigma_n, -\sigma_1, \dots, -\sigma_n, 0, \dots, 0)$, where we have $m-n$ 0's at the end. (To make $m+n$ eigenvalues. Indeed $m+n = n+n+(m-n)$.) The proof of this results is by construction. (The construction can be hinted by the work above in this problem actually.) Below is the proof of this result. Note that once we know this result it is clear that $\|B\|_2 = \|A\|_2$. Indeed (1) the 2-norm is same as the maximum singular value so $\|A\|_2 = \max_i \sigma_i$, and (2), for a symmetric matrix (i.e. B), the 2-norm is same as the maximum absolute value eigenvalue.

Proof of previous claimed result. (Note: proof would have been needed for full credit.)
Let $m \geq n$. Let A be m-by-n. We consider the full singular value decomposition of A. We have

$$A = (U_1 U_2) * \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} * W^*$$

where

(a) U_1 is m-by-n, U_2 is $(m-n)$-by-n, Σ is n-by-n, 0 is $(m-n)$-by-n and W is n-by-n,

(b) $(U_1 U_2)$ is orthogonal: $(U_1 U_2)^* (U_1 U_2) = (U_1 U_2)(U_1 U_2)^* = I_m$,

(c) W is orthogonal: $W^* W = W W^* = I_n$,

(d) Σ is diagonal with singular values of A on diagonal.

Now we form

$$V = \begin{pmatrix} \frac{\sqrt{2}}{2} U_1 & \frac{\sqrt{2}}{2} U_1 & U_2 \\ \frac{\sqrt{2}}{2} W & -\frac{\sqrt{2}}{2} W & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} \Sigma & & \\ & -\Sigma & \\ & & 0 \end{pmatrix}.$$

(Note: the 0 matrix in D is $(m-n)$-by-$(m-n)$.)
We claim (simple computations) that

(a) $B = VDV^*$,

(b) $V^* V = V V^* = I_{m+n}$,

(c) D is diagonal.

So V is the matrix of eigenvectors of B and the eigenvalues of B are in D. A similar result exists for $n \geq m$.

Problem 5

Given the 2-by-2 real matrix $A = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$, determine the set of all real a, b such that

(a) A is orthogonal.

(b) A is symmetric positive definite.

(c) A is nilpotent.

(d) A is unitarily diagonalizable.

Solution:

(a) We want A to be orthogonal. That is $A^T A = AA^T = I$. This means that the row of A has to form an orthonormal basis. The first row of A is $\begin{pmatrix} 0 & 1 \end{pmatrix}$, the second row has to be $\begin{pmatrix} \pm 1 & 0 \end{pmatrix}$. We have

$$A = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}, \quad a = \pm 1, \quad b = 0.$$

(b) We want A to be symmetric positive definite. Clearly by symmetry a has to be 1. Now we know that a symmetric positive definite matrix has a positive diagonal. (Because, we have that for all nonzero vector x, we need $x^T A x > 0$; and we also have that $a_{ii} = e_i^T A e_i$.) And we see that a_{11} is 0 so no value b can make A symmetric positive definite. There is no solution.

(c) We want A to be nilpotent. For A to be nilpotent, we need $A^2 = 0$. We have

$$\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} = \begin{pmatrix} a & b \\ ab & a + b^2 \end{pmatrix}.$$

For this to be the zero matrix, we want $a = 0$ and $b = 0$. We have

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a = 0, \quad b = 0.$$

(d) We want A to be unitarily diagonalizable. This is equivalent to we want A to be normal. For A to be normal, we need $A^T A = AA^T$. We have

$$\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}.$$

This leads to

$$\begin{pmatrix} 1 & b \\ b & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab & 1 + b^2 \end{pmatrix}.$$

So this leads to either($a = 1$ and b is any real number) or ($a = -1$ and $b = 0$). We have

$$\text{either } \left(A = \begin{pmatrix} 0 & 1 \\ 1 & b \end{pmatrix}, \quad a = 1, \quad b \in \mathbb{R} \right) \text{ or } \left(A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

Problem 6

<u>Note:</u> *Question (a) and (b) are not related.*

(a) We consider $\mathcal{C}([0,1])$, the space of continuous function on $[0,1]$. We dot $\mathcal{C}([0,1])$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

We consider the subspace $\mathcal{P}_1$ of all polynomials of degree 1 or less on the unit interval $0 \leq x \leq 1$.

Find the least squares approximation to the function $f(x) = x^3$ by a polynomial $p \in \mathcal{P}_1$ on the interval $[0,1]$, i.e., find $p \in \mathcal{P}_1$ that minimizes $\|p - f\|_2$.

(b) We consider the vector space $\mathcal{P}_2$ of all polynomials of degree 2 or less on the unit interval $0 \leq x \leq 1$. We consider the set of functions

$$S = \{p \in \mathcal{P}_2 : \int_0^1 p(x)dx = \int_0^1 p'(x)dx\}.$$

Show that this is a linear subspace of $\mathcal{P}_2$, determine its dimension and find a basis for $\mathcal{S}$.

Solution:

(a) <u>Goal:</u> We want to find the orthogonal projection p of f onto $\mathcal{P}_1$. (Orthogonal in the sense of the given inner product.)

<u>Method:</u>

(1) Get a basis (e_1, e_2) of $\mathcal{P}_1$,

(2) Construct an orthonormal basis (q_1, q_2) of $\mathcal{P}_1$ from (e_1, e_2) using the Gram-Schmidt procedure

i. $r_{11} = \|e_1\|$,

ii. $q_1 = e_1 / r_{11}$,

iii. $r_{12} = \langle q_1, e_2 \rangle$,

iv. $w = e_2 - r_{12} q_1$,

v. $r_{22} = \|w\|$,

vi. $q_2 = w / r_{22}$.

(3) Compute p, the orthogonal projection of f onto $\mathcal{P}_1$, with the formula

$$p = \langle q_1, f \rangle q_1 + \langle q_2, f \rangle q_2.$$

<u>Computation:</u>

(1) $e_1 = 1$ and $e_2 = x$ is a basis of $\mathcal{P}_1$, let us take this one,

(2) i.

$$\|e_1\|^2 = \langle e_1, e_1\rangle = \int_0^1 dx = 1 \text{ so } r_{11} = \|e_1\| = 1,$$

ii.

$$q_1 = e_1/r_{11} = 1,$$

iii.

$$r_{12} = \langle q_1, e_2\rangle = \int_0^1 x\ dx = \frac{1}{2},$$

iv.

$$w = e_2 - r_{12}q_1 = x - \frac{1}{2},$$

v.

$$r_{22} = \|w\| = \int_0^1 (x - \frac{1}{2})^2\ dx = \frac{1}{12},$$

vi.

$$q_2 = w/r_{22} = \sqrt{3}(2x - 1).$$

$(1, 2\sqrt{3}x - \sqrt{3})$ is an orthonormal basis of $\mathcal{P}_1$.

(3) i.

$$\langle q_1, f\rangle = \int_0^1 x^3\ dx = \frac{1}{4},$$

ii.

$$\langle q_2, f\rangle = \int_0^1 \sqrt{3}(2x-1)x^3\ dx = \frac{3}{20}\sqrt{3},$$

iii.

$$p = \langle q_1, f\rangle q_1 + \langle q_2, f\rangle q_2 = \frac{1}{4} + (\frac{3}{20}\sqrt{3})(\sqrt{3}(2x-1)) = \frac{9}{10}x - \frac{1}{5}.$$

The orthogonal projection of f onto $\mathcal{P}_1$ is $p = \frac{9}{10}x - \frac{1}{5}$. It is also the least squares approximation to the function $f(x) = x^3$ by a polynomial $p \in \mathcal{P}_1$ on the interval $[0, 1]$. That is p is such that it minimizes $\|q - f\|_2$ over all q in $\mathcal{P}_1$.

(b) To prove that S is a linear subspace, one takes two polynomials p and q in S and two real numbers α and β and proves that the polynomial $\alpha p + \beta q$ is in S. This is routine exercise. We skip the writing here.

We now want to find a basis of $\mathcal{S}$ and find its dimension. Let $p \in \mathcal{P}_2$, then p writes

$$p = ax^2 + bx + c.$$

If p is in S, it must sastifies the constraint

$$\int_0^1 p(x)dx = \int_0^1 p'(x)dx,$$

that is

$$\int_0^1 \left(ax^2 + bx + c\right) dx = \int_0^1 (2ax + b)\, dx,$$

that is

$$\frac{1}{3}a + \frac{1}{2}b + c = a + b,$$

that is

$$4a + 3b - 6c = 0.$$

Two linearly independent vectors (a, b, c) satisfying this equation are for example: $(3, 0, 2)$ and $(0, 2, 1)$. So a basis for S is for example

$$\left(\ 3x^2 + 2,\quad 2x + 1\ \right).$$

S is of dimension 2.

Problem 7

Let E, F, and G be vector spaces. Let $f \in \mathcal{L}(E,F)$ and $g \in \mathcal{L}(F,G)$. Prove that:

$$\text{Range}(g \circ f) = \text{Range}(g) \iff \text{Null}(g) + \text{Range}(f) = F$$

Solution:
$\boxed{\Leftarrow}$ We assume that $\text{Null}(g) + \text{Range}(f) = F$. It is clear that $\text{Range}(g \circ f) \subset \text{Range}(g)$. We want to prove that $\text{Range}(g) \subset \text{Range}(g \circ f)$. Let $x \in \text{Range}(g)$. We want to prove that $x \in \text{Range}(g \circ f)$. Since $x \in \text{Range}(g)$, there exists $y \in F$ such that $x = g(y)$. Since $y \in F$ and $F = \text{Null}(g) + \text{Range}(f)$ (by our assumption), there exists $u \in \text{Null}(g)$ and $v \in \text{Range}(f)$ such that $y = u + v$. Since $v \in \text{Range}(f)$, there exists $w \in E$ such that $v = f(w)$. Now we have that $x = g(y)$, $y = u + v$ and $v = f(w)$, so $x = g(u + f(w))$, so $x = g(u) + (g \circ f)(w)$, but $u \in \text{Null}(g)$, so $x = (g \circ f)(w)$, so $x \in \text{Range}(g \circ f)$.

$\boxed{\Rightarrow}$ We assume that $\text{Range}(g \circ f) = \text{Range}(g)$. It is clear that $\text{Null}(g) + \text{Range}(f) \subset F$. We want to prove that $F \subset \text{Null}(g) + \text{Range}(f)$. Let $x \in F$. We want to prove that $x \in \text{Null}(g)+\text{Range}(f)$. So we want to find $u \in \text{Null}(g)$ and $v \in \text{Range}(f)$ such that $x = u+v$. Clearly, we have that $g(x) \in \text{Range}(g)$ so (by our assumption) $g(x) \in \text{Range}(g \circ f)$), so there exists $z \in E$ such that $g(x) = (g \circ f)(z)$. Let $u = x - f(z)$ and $v = f(z)$. We have that (please check)(1) $x = u+v$, (2) $u \in \text{Null}(g)$ and (3) $v \in \text{Range}(f)$. So $x \in \text{Null}(g)+\text{Range}(f)$.

Problem 8

Let A and B be $n \times n$ complex matrices such that $AB = BA$. Show that if A has n distinct eigenvalues, then A, B, and AB are all diagonalizable.

Solution:

This question was given as question #4 in the June 2012 Linear Algebra Preliminary Exam. Please check answer there.