University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
Jan. 12, 2024

Student Number: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete six problems. You are allowed to take a break of up to 45 minutes; please start this break not earlier than 90 minutes and not later than 150 minutes into the exam.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper and leave at least a half-inch margin.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{m\times n}$ are the vector spaces of n-tuples and $m \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

Applied Linear Algebra Preliminary Exam Committee:
Stephen Billups, Steffen Borgwardt (Chair), Julien Langou

Part I. Work **all** of problems 1 through 4.

Problem 1. Let $a \neq 0, b \neq 0 \in \mathbb{R}$ be fixed. The below questions require a case distinction based on values of a and b. Consider the matrix

$$A = \begin{pmatrix} a-b & a+b & a+b & a-b \\ 0 & 0 & a-b & a-b \\ 0 & a-b & 0 & b-a \\ b-a & 0 & 0 & b-a \end{pmatrix}.$$

1. (10 points) Find a basis for $\text{null}(A)$.
2. (5 points) Find a basis for $\text{range}(A)$.
3. (5 points) Find a basis for the subspace $S = \text{null}(A) \cap \text{range}(A)$.

Problem 2. Let V be a finite-dimensional real inner product space. Let $T \in \mathcal{L}(V)$. Let U be a subspace of V that is invariant under T.

1. Show that $U^{\perp}$ is invariant under T^*.
2. Construct an example of a $T \in \mathcal{L}(V)$ with a subspace U for which U is invariant under T but $U^{\perp}$ is not invariant under T. In your answer, give V, T and U, then show that U is invariant under T and show that $U^{\perp}$ is not invariant under T.

Problem 3. Let A be a Hermitian matrix over $\mathbb{C}$ that is positive and invertible. (Such a matrix A is often called "Hermitian positive definite".) And let B be a Hermitian matrix over $\mathbb{C}$.

1. (10 points) Show that there exists an invertible matrix P such that $P^H AP = I$ and $P^H BP$ is diagonal.
   (Hint: First, show that there exists an invertible matrix T such that $A = T^H T$.)

2. (10 points) If B is positive, (such a matrix B is often called "Hermitian positive semidefinite",) show that
$$\det(A + B) \geq \det(A).$$

Problem 4. Let V be a vector space over the field $\mathbb{F}$. For any $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$, let $G(\lambda, T)$ denote the generalized eigenspace of T corresponding to λ. Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that $G(\lambda, T) = G(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.
(Hint: first show that $G(\lambda, T) \subseteq G(\frac{1}{\lambda}, T^{-1})$.)

Part II. Work **two** of problems 5 through 8.

Problem 5. Let $A \in \mathbb{R}^{m \times n}$ with $m \leq n$.

1. (8 points) Prove that A is full rank if and only if AA^T is invertible.
2. (12 points) Let A now be of full rank. Prove that the matrix $P = I - A^T(AA^T)^{-1}A$ is the orthogonal projection matrix of $\mathbb{R}^n$ onto null (A).

Problem 6. Let V be a finite-dimensional inner product space. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $v_1, \ldots, v_n$ are vectors in V such that

$$\|e_j - v_j\| < \frac{1}{\sqrt{n}}$$

for each j. Prove that $v_1, \ldots, v_n$ is a basis of V.

Hint #1: First prove that for any n scalars a_j, $j = 1, \ldots, n$, we have

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} |a_j| \leq \left(\sum_{j=1}^{n} |a_j|^2 \right)^{\frac{1}{2}}.$$

Hint #2: Assume a_1, a_2, ..., a_n are scalars such that $\sum_{j=1}^{n} a_j v_j = 0$ and look at $\left\| \sum_{j=1}^{n} a_j (e_j - v_j) \right\|$.

Problem 7. Let $A, B \in \mathbb{R}^{n \times n}$. Two matrices A, B are called simultaneously diagonalizable if there exists an invertible matrix S such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal.

1. (6 points) Prove that if A, B are simultaneously diagonalizable then $AB = BA$.
2. (14 points) Prove that if $AB = BA$ and if one of the matrices has n distinct eigenvalues then A, B are simultaneously diagonalizable.

Problem 8.

Let A and B in $\mathbb{R}^{n \times n}$ such that $AB - BA = A$.

1. (10 points) Prove that $A^k B - BA^k = kA^k$
2. (10 points) Prove that A is nilpotent.