University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam Solutions
Jan 13, 2023

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- <u>Notation</u>: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

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Part I. Work **all** of problems 1 through 4.

Problem 1. Suppose U, W are subspaces of a finite-dimensional vector space V.

(a) Show that $\dim(U \cap W) = \dim U + \dim W - \dim(U + W)$.

(b) Let $n = \dim V$. Show that if $k < n$ then an intersection of k subspaces of dimension $n-1$ always has dimension at least $n-k$.

Solution

(a) Let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ be a basis for $U \cap W$. Extend it separately to a basis $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_l\}$ of U and $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ of W. Then $\dim U \cap W = k$, $\dim U = k + l$ and $\dim W = k + m$. So it remains to prove that $\dim U + W = k + l + m$. To show this, we will show all the vectors together

$$\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_l, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$$

form a basis for $U + W$.

Let $\boldsymbol{y} \in U + W$. Then $\boldsymbol{y}$ can be written as $\boldsymbol{u} + \boldsymbol{w}$, where $\boldsymbol{u} \in U$ and $\boldsymbol{w} \in W$. Since $\boldsymbol{u}$ can be written as a linear combination of $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_l\}$ and $\boldsymbol{w}$ can be written as a linear combination of $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$, we conclude $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_l, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$ spans $U + W$.

Take scalars $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l, \gamma_1, \ldots, \gamma_m$ such that

$$\alpha_1 \boldsymbol{x}_1 + \cdots + \alpha_k \boldsymbol{x}_k + \beta_1 \boldsymbol{u}_1 + \cdots + \beta_l \boldsymbol{u}_l + \gamma_1 \boldsymbol{w}_1 + \cdots + \gamma_m \boldsymbol{w}_m = \boldsymbol{0}.$$

Note that $\boldsymbol{w} := \gamma_1 \boldsymbol{w}_1 + \cdots + \gamma_m \boldsymbol{w}_m = -(\alpha_1 \boldsymbol{x}_1 + \cdots + \alpha_k \boldsymbol{x}_k + \beta_1 \boldsymbol{u}_1 + \cdots + \beta_l \boldsymbol{u}_l) \in U$. Also, it is clear $\boldsymbol{w} \in W$. So $\boldsymbol{w} \in U \cap W$. Therefore, there are scalars $\mu_1, \ldots, \mu_k$ such that

$$\boldsymbol{w} = \mu_1 \boldsymbol{x}_1 + \cdots + \mu_k \boldsymbol{x}_k$$

As a result,

$$\gamma_1 \boldsymbol{w}_1 + \cdots + \gamma_m \boldsymbol{w}_m - \mu_1 \boldsymbol{x}_1 - \cdots - \mu_k \boldsymbol{x}_k = \boldsymbol{0}$$

Since $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ is linearly independent, $\gamma_1, \ldots, \gamma_m, \mu_1, \ldots, \mu_k$ are all zeros. Further, we can conclude $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l$ are all zeros. So

$$\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_l, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$$

is a linearly independent set, and hence forms a basis for $U + W$.

(b) We prove it by induction.

If $k = 1$, the result is trivial. Suppose the result holds for some $k \geq 1$. Let $V_1, \dots, V_k, V_{k+1}$ be subspaces of V of dimension $n-1$. Then

$$\dim(\cap_{i=1}^{k+1} V_i) = \dim(V_{k+1} \cap (\cap_{i=1}^{k} V_i)) = \dim V_{k+1} + \dim(\cap_{i=1}^{k} V_k) - \dim(V_{k+1} + \cap_{i=1}^{k} V_i)$$

Note that $\dim(V_{k+1} + \cap_{i=1}^{k} V_i)$ has dimension at most n, V_{k+1} has dimension $n-1$ and by the inductive hypothesis, $\cap_{i=1}^{k} V_i$ has dimension at least $n-k$. Then

$$\dim(\cap_{i=1}^{k+1} V_i) \geq n - 1 + n - k - n = n - (k+1)$$

This completes the proof.

Problem 2.

(a) For each pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{C}^3$, assign a scalar $(\boldsymbol{x}, \boldsymbol{y})$ as follows:

$$(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^* \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \boldsymbol{x}.$$

where $\boldsymbol{y}^*$ is the conjugate transpose of $\boldsymbol{y}$. Is $(\cdot, \cdot)$ an inner product on $\mathbb{C}^3$?

(b) Let V be an inner product space and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$. Prove or disprove

(i) $\|\boldsymbol{u} + \boldsymbol{v}\| \leq \|\boldsymbol{u} + \boldsymbol{w}\| + \|\boldsymbol{w} + \boldsymbol{v}\|$;

(ii) $|\langle \boldsymbol{u}, \boldsymbol{v} \rangle| \leq |\langle \boldsymbol{u}, \boldsymbol{w} \rangle| + |\langle \boldsymbol{w}, \boldsymbol{v} \rangle|$.

Solution

(a) positivity and definiteness: for $\boldsymbol{x} = (x_1, x_2, x_3)^T$, $(\boldsymbol{x}, \boldsymbol{x}) = 2|x_2|^2 + |x_1|^2 + \bar{x_1} x_3 + x_1 \bar{x}_3 + 2|x_3|^2$. Since $|x_1|^2 + \bar{x_1} x_3 + x_1 \bar{x}_3 + |x_3|^2 = (x_1 + x_3)\overline{(x_1 + x_3)} \geq 0$, $(\boldsymbol{x}, \boldsymbol{x}) \geq 0$, and the equality holds if and only if $\boldsymbol{x} = \boldsymbol{0}$.

additivity in first slot: $(\boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{z}^* A(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{z}^* A\boldsymbol{x} + \boldsymbol{z}^* A\boldsymbol{y} = (\boldsymbol{x}, \boldsymbol{z}) + (\boldsymbol{y}, \boldsymbol{z})$ where A denotes the 3×3 matrix.

homogeneity in first slot: $(\lambda \boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^* A \lambda \boldsymbol{x} = \lambda \boldsymbol{y}^* A \boldsymbol{x} = \lambda(\boldsymbol{x}, \boldsymbol{y})$

conjugate symmetry: $\overline{(\boldsymbol{x}, \boldsymbol{y})} = \overline{\boldsymbol{y}^* A \boldsymbol{x}} = \boldsymbol{x}^* A^* \boldsymbol{y} = \boldsymbol{x}^* A \boldsymbol{y} = (\boldsymbol{y}, \boldsymbol{x})$. So it is an inner product.

(b) (i) False. Take $\boldsymbol{u} = \boldsymbol{v}$, $\boldsymbol{w} = -\boldsymbol{u}$.

(ii) False. Consider the standard inner product on $\mathbb{R}^2$. Consider the counterexample: $\boldsymbol{u} = \boldsymbol{v} = (1,1)$ and $\boldsymbol{w} = (1,-1)$.

Problem 3. Let T be a positive operator on a complex inner product space V and S be an operator on V such that $ST = -TS$. Show that $ST = TS = 0$.

Solution Because T is a positive operator on a complex inner product space it is self-adjoint and has only non-negative eigenvalues (*Axler* 7.27). Therefore, by Complex Spectral theorem V has a basis consisting of eigenvectors of T. Let v be an eigenvector of T with eigenvalue λ. Then $T(Sv) = -STv = -\lambda Sv$. If $Sv \neq 0$ and $\lambda > 0$ then Sv would be an eigenvector of T with eigenvalue $-\lambda < 0$, which is impossible. Therefore, $\lambda = 0$ or $Sv = 0$; in either case $TSv = -\lambda Sv = 0$. Because $(TS)v = 0$ for all basis vectors, it is true for any vector of V and therefore $TS = 0$ and $ST = -TS = 0$.

Problem 4. Let V be a vector space over a field $\mathbb{F}$. Suppose $T \in \mathcal{L}(V)$ has minimal polynomial $p(z) = 3 + 2z - z^2 + 5z^3 + z^4$.

(a) (5 pts) Prove that T is invertible.

(b) (15 pts) Find the minimal polynomial of T^{-1}.

Solution

1. T is invertible if and only if 0 is not an eigenvalue of T. Since $p(0) = 3$, 0 is not an eigenvalue of T, so T is invertible.

2. By the definition of the minimal polynomial,
$$0 = p(T) = 3I + 2T - T^2 + 5T^3 + T^4.$$
Muliplying both sides by $\frac{1}{3}T^{-4}$ gives
$$0 = T^{-4} + \frac{2}{3}T^{-3} - \frac{1}{3}T^{-2} + \frac{5}{3}T^{-1} + \frac{1}{3}I.$$
Thus, $q(z) = z^4 + \frac{2}{3}z^3 - \frac{1}{3}z^2 + \frac{5}{3}z + \frac{1}{3}$ is a monic polynomial such that $q(T^{-1}) = 0$. To show that q is the minimal polynomial of T^{-1}, we need to show that there is no nonzero polynomial $r(z)$ of smaller degree such that $r(T^{-1}) = 0$. Suppose such a polynomial $r(z)$ exists, with degree $m < 4$. Define $s(z) = z^m r(1/z)$. Then
$$s(T) = T^m r(T^{-1}) = 0.$$

Thus, $s(z)$ is a nonzero polynomial of degree at most $m < 4$ such that $s(T) = 0$, which contradicts the statement that p is the minimal polynomial of T. Thus, no such polynomial $r(z)$ exists.

It follows that q is the monic polynomial of smallest degree such that $q(T^{-1}) = 0$. Hence q is the minimal polynomial of T^{-1}.

Part II. Work **two** of problems 5 through 8.

Problem 5. Suppose A is a normal matrix such that $A^5 = A^4$.

(a) (8 pts) Prove that A is self-adjoint.

(b) (5 pts) Give a counterexample to Part (a) if A is not normal.

(c) (7 pts) Prove or disprove that A is a projection matrix. (Recall that a matrix A is a projection matrix if $A^2 = A$.)

Solution:

(a) Suppose that A is an $n \times n$ normal matrix. Since A is normal, it has an orthogonal set of n eigenvectors $\{v_1, \dots, v_n\}$. Let λ_i be the eigenvalue associated with v_i. Then

$$\lambda^5 v_i = A^5 v_i = A^4 v_i = \lambda^4 v_i.$$

Since $v_i \neq 0$, $\lambda_i = 0$ or 1. Since A is normal with real eigenvalues, it is self-adjoint.

(b) A counter example is

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Note that $A^n = A$, for any $n > 0$. However, $AA^T \neq A^T A$, and $A^T \neq A$.

(c) Furthermore, $\lambda_i^2 = \lambda_i$ (since $\lambda_i = 0$ or 1). Thus,

$$A^2 v_i = \lambda_i^2 v_i = \lambda v_i = A v_i.$$

Since $A^2 v_i = A v_i$ for all v_i in basis of n vectors, it follows that $A^2 = A$, so A is a projection matrix.

Problem 6. Let V be a finite-dimensional inner product space over $\mathbb{C}$. Let T be a normal operator on V. Let $\lambda \in \mathbb{C}$ and let $v \in V$ be a unit vector (i.e. $\|v\| = 1$). Prove that T has an eigenvalue λ' such that

$$\|\lambda - \lambda'\| \leq \|Tv - \lambda v\|.$$

Solution: Since T is normal, V has an orthonormal basis $(v_1, \dots, v_n)$ consisting of eigenvectors of T. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. Using this basis, we can write $v = a_1 v_1 + \cdots + a_n v_n$ for some scalars $a_1, \dots, a_n \in \mathbb{C}$. Hence,

$$\begin{aligned}
\|Tv - \lambda v\|^2 &= \|T(a_1 v_1 + \cdots a_n v_n) - \lambda(a_1 v_1 + \cdots a_n v_n)\|^2 \\
&= \|(\lambda_1 - \lambda)a_1 v_1 + \cdots + (\lambda_n - \lambda)a_n v_n\|^2 \\
&= |\lambda_1 - \lambda|^2 |a_1|^2 + \cdots + |\lambda_n - \lambda|^2 |a_n|^2 \quad \text{(since } v_i\text{s are orthonormal)} \\
&\geq \min_i |\lambda_i - \lambda|^2 (|a_1|^2 + \cdots |a_n|^2) \\
&= \min_i |\lambda_i - \lambda|^2 \, \|v\|^2 \\
&= \min_i |\lambda_i - \lambda|^2 \\
&= |\lambda_j - \lambda|^2 \text{ for some } j.
\end{aligned}$$

Thus, for some eigenvalue λ_j, we have

$$\|\lambda - \lambda_j\| \leq \|Tv - \lambda v\|.$$

Problem 7. Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ be two sets of vectors of an inner product space V of dimension n. Suppose

$$\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle, \quad i, j = 1, 2, \dots, n.$$

(a) Let $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_t\}$, $t \leq n$, be a basis for span $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$. Show that $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_t\}$ is a basis for span $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$.

(b) Show that there exists an isometry S on V such that

$$S(\boldsymbol{u}_i) = \boldsymbol{v}_i, \quad i = 1, 2, \dots, n.$$

Solution

(a) First we show $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_t\}$ is linearly independent. Let

$$\alpha_1 \boldsymbol{v}_1 + \cdots \alpha_t \boldsymbol{v}_t = \mathbf{0}$$

Then $0 = \langle \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_t \boldsymbol{v}_t, \boldsymbol{v}_i \rangle = \langle \alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_t \boldsymbol{u}_t, \boldsymbol{u}_i \rangle$ for all $i \leq t$, which means $\alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_t \boldsymbol{u}_t$ is orthogonal to all basis vectors, so $\alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_t \boldsymbol{u}_t = \mathbf{0}$. Due to the linear indepdence of $\boldsymbol{u}_1, \dots, \boldsymbol{u}_t$, we have $\alpha_1 = \cdots = \alpha_t = 0$.

Now we show that $\dim(\operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}) \leq t$. For any $\boldsymbol{v} \in V$, $\boldsymbol{v} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n$, so

$$\begin{aligned}\langle \boldsymbol{v}, \boldsymbol{v}_i \rangle &= \langle \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n, \boldsymbol{v}_i \rangle = \langle \alpha_1 \boldsymbol{u}_1 + \cdots + \alpha_n \boldsymbol{u}_n, \boldsymbol{u}_i \rangle \\ &= \langle \beta_1 \boldsymbol{u}_1 + \cdots + \beta_t \boldsymbol{u}_t, \boldsymbol{u}_i \rangle = \langle \beta_1 \boldsymbol{v}_1 + \cdots + \beta_t \boldsymbol{v}_t, \boldsymbol{v}_i \rangle\end{aligned}$$

for all $i = 1, \dots, n$. Therefore

$$\langle \boldsymbol{v} - (\beta_1 \boldsymbol{v}_1 + \cdots + \beta_t \boldsymbol{v}_t), \boldsymbol{v}_i \rangle = 0$$

for all $i = 1, \dots, n$, which means $\boldsymbol{v} = \beta_1 \boldsymbol{v}_1 + \cdots + \beta_t \boldsymbol{v}_t$. So the dimension of V is no greater than t. Combining the fact that $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_t\}$ is linearly independent, we can conclude $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_t\}$ is a basis for $\operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$.

(b) Without loss of generality, let $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_t\}$ be a basis for $\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$. Then by Part (a), $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_t\}$ is a basis for $\operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_n\}$. Now let $\{\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-t}\}$ and $\{\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n-t}\}$ be orthonormal bases for $(\operatorname{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_t\})^\perp$ and $(\operatorname{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_t\})^\perp$, respectively. Then

$$\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_t, \boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_{n-t}\} \text{ and } \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_t, \boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_{n-t}\}$$

are two bases for V. Now suppose $\boldsymbol{x} \in V$. If

$$\boldsymbol{x} = \sum_{i=1}^{t} x_i \boldsymbol{u}_i + \sum_{i=1}^{n-t} y_i \boldsymbol{\alpha}_i,$$

then the linear map S (constructed based on Theorem 3.5 of Axler), defined by

$$S\boldsymbol{x} = \sum_{i=1}^{t} x_i \boldsymbol{v}_i + \sum_{i=1}^{n-t} y_i \boldsymbol{\beta}_i$$

is the isometry wanted.

To verify it, first notice $S(\boldsymbol{u}_i) = S(1\boldsymbol{u}_i) = 1\boldsymbol{v}_i = \boldsymbol{v}_i$. Second, we have

$$\begin{aligned}\|S\boldsymbol{x}\| = \sqrt{\langle S\boldsymbol{x}, S\boldsymbol{x}\rangle} &= \sqrt{\sum_{i=1}^{t}\sum_{j=1}^{t} x_i \bar{x}_j \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle + \sum_{i=1}^{n-t} y_i \bar{y}_i \langle \boldsymbol{\beta}_i, \boldsymbol{\beta}_i \rangle} \\ &= \sqrt{\sum_{i=1}^{t}\sum_{j=1}^{t} x_i \bar{x}_j \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle + \sum_{i=1}^{n-t} y_i \bar{y}_i} \\ &= \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \|\boldsymbol{x}\|.\end{aligned}$$

Problem 8. Let V be a real inner product space and P a projection operator on V, $P^2 = P$. Prove that operator $I - 2P$ is an isometry if and only if P is self-adjoint.

Solution:

For all $x \in V$, $\langle (I-2P)x, (I-2P)x \rangle = ||x||^2 - 2\langle Px, x\rangle - 2\langle x, Px\rangle + 4\langle Px, Px\rangle$. If P is self-adjoint, $\langle Px, Px\rangle = \langle P^2x, x\rangle = \langle Px, x\rangle$ and $\langle (I-2P)x, (I-2P)x\rangle = ||x||^2 - 2\langle Px, x\rangle - 2\langle x, Px\rangle + 4\langle Px, x\rangle = ||x||^2 + 2\langle Px, x\rangle - 2\langle x, Px\rangle = ||x||^2$. The last equality uses the self-adjoint property as well. Thus, $I - 2P$ is an isometry.

Conversely, suppose $I - 2P$ is an isometry. Then for all $x \in V$, $||(I-2P)x||^2 = ||x||^2$, so

$$\langle (I-2P)x, (I-2P)x\rangle = ||x||^2 - 2\langle Px, x\rangle - 2\langle x, Px\rangle + 4\langle Px, Px\rangle = ||x||^2,$$

and therefore

$$2\langle Px, Px\rangle = \langle Px, x\rangle + \langle x, Px\rangle .$$

We can now show that P is self-adjoint. Define indices 1 and 2 such that $x_1 = Px$ and $x_2 = (I-P)x$. For any two vectors $y, z \in V$, consider $x = y_1 + z_2$ so that $Px = y_1$ and $(I-P)x = z_2$ and substitute it into the equation above

$$2\langle y_1, y_1\rangle = \langle y_1, y_1 + z_2\rangle + \langle y_1 + z_2, y_1\rangle = 2\langle y_1, y_1\rangle + \langle y_1, z_2\rangle + \langle z_2, y_1\rangle .$$

Therefore, $\langle y_1, z_2\rangle + \langle z_2, y_1\rangle = 0$ and $\langle y_1, z_2\rangle = -\langle z_2, y_1\rangle = 0$ because real inner product is symmetric.

Finally, one concludes that

$$\langle y, Pz\rangle = \langle y_1 + y_2, z_1\rangle = \langle y_1, z_1\rangle = \langle y_1, z_1 + z_2\rangle = \langle Py, z\rangle .$$

Since this is true for all $y, z \in V$, P is self-adjoint.
