University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam Solutions
Aug. 11, 2023

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Please begin each problem on a new page, and write the problem number and page number at the top of each page. (For example, 6-1, 6-2, 6-3 for pages 1, 2 and 3 of problem 6). Please write only on one side of the paper.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do not submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

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Part I. Work **all** of problems 1 through 4.

Problem 1. Let T be a linear map $T : U \to V$ and S be a linear map $S : V \to W$. Prove that $\dim U - \dim V \leq \dim \text{null } ST - \dim \text{null } S$.

Solution By the rank-nullity theorem $\dim U = \dim \text{null } ST + \dim \text{range } ST$ and $\dim V = \dim \text{null } S + \dim \text{range } S$. Subtracting right and left sides one has $\dim U - \dim V = \dim \text{null } ST - \dim \text{null } S + (\dim \text{range } ST - \dim \text{range } S)$. But range ST is a subspace of range S, and therefore $\dim \text{range } ST \leq \dim \text{range } S$ and thus $\dim U - \dim V \leq \dim \text{null } ST - \dim \text{null } S$.

Problem 2.

Let $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ be 3 unit vectors in a real inner-product space V.

(a) (15 points) Show that $2\langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle \langle \mathbf{v}, \mathbf{w}\rangle \geq \langle \mathbf{u}, \mathbf{v}\rangle^2 + \langle \mathbf{u}, \mathbf{w}\rangle^2 + \langle \mathbf{v}, \mathbf{w}\rangle^2 - 1$. Hint: apply the first step of the Gram-Schmidt process to vectors $\mathbf{v}$ and $\mathbf{w}$ with respect to $\mathbf{u}$ and and apply the Cauchy-Schwarz inequality to the resulting pair of vectors.

(b) (5 points) Show that the equality is reached if and only if vectors $\mathbf{u}, \mathbf{v},$ and $\mathbf{w}$ are linearly dependent.

Solution

(a) Apply Cauchy-Schwarz inequality to vectors $\mathbf{v} - \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}$ and $\mathbf{w} - \langle \mathbf{u}, \mathbf{w}\rangle \mathbf{u}$

$$\langle \mathbf{v} - \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}, \mathbf{w} - \langle \mathbf{u}, \mathbf{w}\rangle \mathbf{u}\rangle^2 \leq ||\mathbf{v} - \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}||^2 ||\mathbf{w} - \langle \mathbf{u}, \mathbf{w}\rangle \mathbf{u}||^2$$

The inner product on the left can be written as

$$\langle \mathbf{v} - \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}, \mathbf{w} - \langle \mathbf{u}, \mathbf{w}\rangle \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle - \langle \mathbf{u}, \mathbf{w}\rangle \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle = \langle \mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle$$

and the squares of the norms on the right can be written as

$$||\mathbf{v} - \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}||^2 = \langle \mathbf{v} - \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}, \mathbf{v} - \langle \mathbf{u}, \mathbf{v}\rangle \mathbf{u}\rangle = \langle \mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{v}\rangle = 1 - \langle \mathbf{u}, \mathbf{v}\rangle^2$$

and similarily $||\mathbf{w} - \langle \mathbf{u}, \mathbf{w}\rangle \mathbf{u}||^2 = 1 - \langle \mathbf{u}, \mathbf{w}\rangle^2$.
Cauchy-Schwarz inequality gives

$$(\langle \mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle \langle \mathbf{u}, \mathbf{w}\rangle)^2 \leq (1 - \langle \mathbf{u}, \mathbf{v}\rangle^2)(1 - \langle \mathbf{u}, \mathbf{w}\rangle^2).$$

Expansions on both sides lead to

$$\langle \mathbf{v}, \mathbf{w} \rangle^2 - 2 \langle \mathbf{v}, \mathbf{w} \rangle \langle \mathbf{u}, \mathbf{v} \rangle \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle^2 \langle \mathbf{u}, \mathbf{w} \rangle^2 \leq 1 - \langle \mathbf{u}, \mathbf{v} \rangle^2 - \langle \mathbf{u}, \mathbf{w} \rangle^2 + \langle \mathbf{u}, \mathbf{v} \rangle^2 \langle \mathbf{u}, \mathbf{w} \rangle^2,$$

which leads to the desired inequality after subtraction of $\langle \mathbf{v}, \mathbf{w} \rangle^2 + \langle \mathbf{u}, \mathbf{v} \rangle^2 \langle \mathbf{u}, \mathbf{w} \rangle^2$ and a sign change.

(b) Cauchy-Schwarz inequality becomes equality if and only if the two vectors are linearly dependent. This means, WLOG, that $\mathbf{v} - \langle \mathbf{u}, \mathbf{v} \rangle \mathbf{u} = a(\mathbf{w} - \langle \mathbf{u}, \mathbf{w} \rangle \mathbf{u})$ for some real a. Therefore, equality is reached if and only if $\mathbf{v} = (\langle \mathbf{u}, \mathbf{v} \rangle - a \langle \mathbf{u}, \mathbf{w} \rangle)\mathbf{u} + a\mathbf{w}$, which proves linear dependence. In the other direction, if $\mathbf{v} = b\mathbf{u} + a\mathbf{w}$ then $\langle \mathbf{u}, \mathbf{v} \rangle = b + a \langle \mathbf{u}, \mathbf{w} \rangle$ and thus $b = \langle \mathbf{u}, \mathbf{v} \rangle - a \langle \mathbf{u}, \mathbf{w} \rangle$.

Problem 3. Let T be a positive operator on V. Suppose $v, w \in V$ are such that $Tv = w$ and $Tw = v$. Prove that $v = w$.

Solution Since T is a positive operator, we have $\langle T(v-w), v-w \rangle \geq 0$. On the other hand, since $Tv = w$ and $Tw = v$, we have $T(v-w) = w - v$. Thus

$$0 \leq \langle T(v-w), v-w \rangle = \langle w-v, v-w \rangle = -\|v-w\|^2 \leq 0.$$

Therefore $\|v - w\| = 0$, so $v = w$.

Problem 4. Let $n \geq 2$.

(a) Is there an $n \times n$ matrix A with $A^{n-1} \neq 0$ and $A^n = 0$? Give an example to show such a matrix exists (and explain why the matrix satisfies the two conditions), or disprove it.

(b) Show that an $n \times n$ upper triangular matrix A with $A^n \neq 0$ and $A^{n+1} = 0$ does not exist.

Solution

1. Yes. For example, let A be the matrix such that $A_{i,i+1} = 1$ for $i = 1, \ldots, n-1$ and $A_{ij} = 0$ otherwise. The matrix is already in Jordan canonical form and 0 is the only eigenvalue. The largest Jordan block corresponding to 0 is n, so the minimal polynomial is $p(x) = x^n$. Therefore we conclude $A^{n-1} \neq 0$, and $A^n = 0$.

2. Suppose $A^{n+1} = 0$. Let λ be an eigenvalue of A (λ exists due to A being upper triangular) with nonzero eigenvector $\boldsymbol{v}$. Then

$$A\boldsymbol{v} = \lambda\boldsymbol{v} \Rightarrow A^2\boldsymbol{v} = \lambda A\boldsymbol{v} = \lambda^2\boldsymbol{v} \Rightarrow \cdots \Rightarrow A^{n+1}\boldsymbol{v} = \lambda^n A\boldsymbol{v} = \lambda^{n+1}\boldsymbol{v}.$$

However, $A^{n+1} = 0$, so $\lambda^{n+1}\boldsymbol{v} = \mathbf{0}$, which implies $\lambda = 0$. Thus all eigenvalues of A are zero. Then the minimal polynomial of A, $p(x)$, has only zero as a root and thus $p(x) = x^k$, $k \leq n$. Therefore, $p(A) = 0 \Rightarrow A^k = 0 \Rightarrow A^n = 0$.

Part II. Work **two** of problems 5 through 8.

Problem 5. Let T be a linear map on a vector space V, $\dim V = n$.

(a) If for some vector $\boldsymbol{v}$, the vectors $\boldsymbol{v}$, $T(\boldsymbol{v})$, $T^2(\boldsymbol{v})$, $\dots$, $T^{n-1}(\boldsymbol{v})$ are linearly independent, show that every eigenvalue of T has only one corresponding eigenvector up to a scalar multiplication.

(b) If T has n distinct eigenvalues, and vector $\boldsymbol{u}$ is the sum of n eigenvectors, corresponding to the distinct eigenvalues, show that $\boldsymbol{u}$, $T(\boldsymbol{u})$, $T^2(\boldsymbol{u})$, $\dots$, $T^{n-1}(\boldsymbol{u})$ are linearly independent (and thus form a basis of V).

Solution:

(a) The vectors $\boldsymbol{v}$, $T(\boldsymbol{v})$, $T^2(\boldsymbol{v})$, $\dots$, $T^{n-1}(\boldsymbol{v})$ form a basis for V. The matrix representation of the linear map under this basis has a matrix whose first $n-1$ columns have a subdiagonal of 1's and 0's elsewhere. Therefore, for any eigenvalue λ, the matrix $A - \lambda I$ has a rank of $n-1$. Based on the rank-nullity theorem, we know that $\dim \operatorname{null} (A - \lambda I) = n - (n-1) = 1$, which means the eigenvectors belonging to λ are multiples of each other.

(b) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ be eigenvectors (that form a basis) corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of T. Let $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \cdots + \boldsymbol{u}_n$. Then $T(\boldsymbol{u}) = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_n u_n$, $T^2(\boldsymbol{u}) = \lambda_1^2 u_1 + \lambda_2^2 u_2 + \cdots + \lambda_n^2 u_n, \dots$, $T^{n-1}(\boldsymbol{u}) = \lambda_1^{n-1} u_1 + \lambda_2^{n-1} u_2 + \cdots + \lambda_n^{n-1} u_n$. The coefficient matrix of $\boldsymbol{u}$, $T(\boldsymbol{u})$, $\dots$, $T^{n-1}\boldsymbol{u}$ under the basis $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\dots$, $\boldsymbol{u}_n$ is a Vandermonde matrix, which is invertible for distinct $\lambda_1, \dots, \lambda_n$ (it can be easily shown that the columns of the matrix are linearly independent).

Problem 6. Let A be an $n \times n$ positive semidefinite matrix.

(a) Show that
$$\left\|(I - A)(I + A)^{-1}\boldsymbol{x}\right\|_2 \le \|\boldsymbol{x}\|_2, \ \boldsymbol{x} \in \mathbb{C}^n.$$

(b) Show that $\boldsymbol{x} \in \operatorname{null} A$ is equivalent to
$$(I - A)(I + A)^{-1}\boldsymbol{x} = \boldsymbol{x}.$$

Solution:

(a) To show $\left\|(I-A)(I+A)^{-1}\boldsymbol{x}\right\|_2 \le \|\boldsymbol{x}\|_2$, it is sufficient to show

$$\boldsymbol{x}^*(I+A)^{-1}(I-A)(I-A)(I+A)^{-1}\boldsymbol{x} \le \boldsymbol{x}^*\boldsymbol{x},\ \boldsymbol{x} \in \mathbb{C}^n,$$

which is equivalent to showing $I-(I+A)^{-1}(I-A)(I-A)(I+A)^{-1} = I-(I+A)^{-1}(I-A)^2(I+A)^{-1}$ is positive semidefinite, which is further equivalent to showing

$$(I+A)I(I+A)-(I+A)(I+A)^{-1}(I-A)^2(I+A)^{-1}(I+A) = (I+A)^2-(I-A)^2$$

is positive semidefinite. But, $(I+A)^2-(I-A)^2 = 4A$, which is positive semidefinite.

(b) Next we show the equivalence of $\boldsymbol{x} \in \text{null } A$ and $(I-A)(I+A)^{-1}\boldsymbol{x} = \boldsymbol{x}$. If $\boldsymbol{x} \in \text{null } A$, then $A\boldsymbol{x} = \boldsymbol{0}$. Hence $(I-A)\boldsymbol{x} = \boldsymbol{x}-A\boldsymbol{x} = \boldsymbol{x}$, and $(I+A)\boldsymbol{x} = \boldsymbol{x}+A\boldsymbol{x} = \boldsymbol{x}$, the latter implying $(I+A)^{-1}\boldsymbol{x} = \boldsymbol{x}$ (note that I+A is invertible). Therefore, we have $(I-A)(I+A)^{-1}\boldsymbol{x} = \boldsymbol{x}$.

On the other hand, suppose $(I-A)(I+A)^{-1}\boldsymbol{x} = \boldsymbol{x}$. Since $I+A$ and $I-A$ commute, $(I+A)^{-1}$ and $I-A$ commute. Hence $(I-A)(I+A)^{-1}\boldsymbol{x} = (I+A)^{-1}(I-A)\boldsymbol{x} = \boldsymbol{x}$, or $(I-A)\boldsymbol{x} = (I+A)\boldsymbol{x}$. This implies that $A\boldsymbol{x} = \boldsymbol{0}$.

Problem 7. Let A be an isometry on a finite-dimensional real inner product space V which satisfies $A^2 = -I$. Prove that for every vector $\mathbf{v}$ in V, $A\mathbf{v}$ is orthogonal to $\mathbf{v}$.

Solution For any non-zero $\mathbf{v} \in V$ consider $A\mathbf{v} = a\mathbf{v} + \mathbf{w}$ where a is a scalar and $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. $A^2\mathbf{v} = A(a\mathbf{v}+\mathbf{w}) = a^2\mathbf{v} + a\mathbf{w} + A\mathbf{w} = -\mathbf{v}$. The last equality can be rewritten as $A\mathbf{w} = -a\mathbf{w} - (1+a^2)\mathbf{v}$. Because A is an isometry

$$||A\mathbf{v}||^2 = ||a\mathbf{v}+\mathbf{w}||^2 = a^2||\mathbf{v}||^2 + ||\mathbf{w}||^2 = ||\mathbf{v}||^2,$$

where we have used $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. Thus, $||\mathbf{w}||^2 = (1-a^2)||\mathbf{v}||^2$. Similarily for $A\mathbf{w}$

$$||A\mathbf{w}||^2 = ||-a\mathbf{w} - (1+a^2)\mathbf{v}||^2 = a^2||\mathbf{w}||^2 + (1+a^2)^2||\mathbf{v}||^2 = ||\mathbf{w}||^2$$

and thus, $(1+a^2)^2||\mathbf{v}||^2 = (1-a^2)||\mathbf{w}||^2 = (1-a^2)^2||\mathbf{v}||^2$. Because $||\mathbf{v}||^2 > 0$ it follows that $1+a^2 = |1-a^2|$, which is possible only when $a = 0$. Therefore, $A\mathbf{v} = \mathbf{w}$ and is orthogonal to $\mathbf{v}$.

An alternative solution (credit to Andrew Kitterman) Since A is an isometry, $AA^* = A^*A = I$. We also know that $A^2 + I = 0$. Now, let $\boldsymbol{v} \in V$, and $\boldsymbol{v} \neq \boldsymbol{0}$. Then we have

$$
\begin{aligned}
&\left\langle (A^2+I)\boldsymbol{v}, A\boldsymbol{v}\right\rangle = 0 \quad (\text{as } A^2+I=0)\\
&\to \left\langle A^2\boldsymbol{v}+\boldsymbol{v}, A\boldsymbol{v}\right\rangle = 0\\
&\to \left\langle A^2\boldsymbol{v}, A\boldsymbol{v}\right\rangle + \langle \boldsymbol{v}, A\boldsymbol{v}\rangle = 0\\
&\to \langle A\boldsymbol{v}, A^*A\boldsymbol{v}\rangle + \langle A\boldsymbol{v}, \boldsymbol{v}\rangle = 0\\
&\to \langle A\boldsymbol{v}, \boldsymbol{v}\rangle + \langle A\boldsymbol{v}, \boldsymbol{v}\rangle = 0 \quad (\text{as } AA^*=I)\\
&\to 2\langle A\boldsymbol{v}, \boldsymbol{v}\rangle = 0\\
&\to \langle A\boldsymbol{v}, \boldsymbol{v}\rangle = 0
\end{aligned}
$$

Since $\boldsymbol{v} \neq \mathbf{0}$, this implies $A\boldsymbol{v}$ is orthogonal to $\boldsymbol{v}$ for every $\boldsymbol{v} \in V$.

Problem 8. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be a set of vectors in a real inner-product space such that $\langle \mathbf{v}_i, \mathbf{v}_j\rangle < 0$ for all $i \neq j$.

(a) (5 points) Show that any linear combination of a set of vectors can be written as a difference of two linear combinations with non-negative coefficients.

(b) (7 points) If set S is linearly dependent, show that any nontrivial linear combination of vectors from S equal to $\mathbf{0}$ contains only coefficients of the same sign (disregarding zeros).

(c) (8 points) Show that $\dim \operatorname{span} S \geq n-1$.

Solution

(a) For any linear combination $\sum a_i\mathbf{v}_i$ define

$$
b_i = \begin{cases} a_i, \text{if } a_i \geq 0 \\ 0, \text{if } a_i < 0 \end{cases} \qquad \text{and} \qquad c_i = \begin{cases} -a_i, \text{if } a_i < 0 \\ 0, \text{if } a_i \geq 0 \end{cases}
$$

Then, $b_i \geq 0$ and $c_i \geq 0$ and $\sum a_i\mathbf{v}_i = \sum b_i\mathbf{v}_i - \sum c_i\mathbf{v}_i$ is the required difference of linear combinations.

(b) For a linearly dependent set S there is a non-trivial linear combination $\sum a_i\mathbf{v}_i = \mathbf{0}$. Introduce linear combinations $\sum b_i\mathbf{v}_i$ and $\sum c_i\mathbf{v}_i$ as in part (a). Define $\mathbf{w} = \sum b_i\mathbf{v}_i = \sum c_i\mathbf{v}_i$. Now,

$$
||\mathbf{w}||^2 = \left\langle \sum b_i\mathbf{v}_i, \sum c_j\mathbf{v}_j\right\rangle = \sum\sum b_ic_j\langle \mathbf{v}_i, \mathbf{v}_j\rangle \leq 0
$$

because each term in the last sum is non-positive with $b_i c_j \geq 0$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle < 0$. If both linear combinations $\sum b_i \mathbf{v}_i$ and $\sum c_i \mathbf{v}_i$ are non-trivial than there is at least one product $b_i c_j \neq 0$ and therefore $||\mathbf{w}||^2 < 0$. Contradiction. Therefore, either $\sum b_i \mathbf{v}_i$ or $\sum c_i \mathbf{v}_i$ is trivial and therefore linear combination $\sum a_i \mathbf{v}_i$ has only coefficients of the same sign.

(c) If $\dim \operatorname{span} \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}\} \leq n-2$, then there is a non-trivial linear combination $\sum_{i=1}^{n-1} a_i \mathbf{v}_i = \mathbf{0}$, which according to part (b) has only terms of the same sign and which WLOG can be taken as non-negative. Now,

$$0 = \left\langle \sum_{i=1}^{n-1} a_i \mathbf{v}_i, \mathbf{v}_n \right\rangle = \sum_{i=1}^{n-1} a_i \langle \mathbf{v}_i, \mathbf{v}_n \rangle < 0$$

because all terms of the last sum are non-positive with at least one negative term. Contradiction. Thus, $\dim \operatorname{span} \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}\} = n - 1$, which implies $\dim \operatorname{span} S \geq n - 1$.
