Part I. Work all of problems 1 through 4.

Problem 1. Let $A$ and $B$ be two real $10 \times 10$ matrices. Suppose that the rank of $A$ is 6 and the rank of $B$ is 4. Justify your answers to the following questions.

(a) What is the minimum possible rank of the matrix $A^2$?

(b) What is the maximum possible rank of the matrix $AB^T$?

(c) If the columns of $A$ are orthogonal to the columns of $B$, must the rank of $A + B$ be equal to 10?

Solution

(a) Note that $\text{null } A^2 = \text{null } A + W$, where $W = \{x \in \text{Row } A | Ax \in \text{null } A \}$. Consider the mapping $T : \text{Row } A \to \mathbb{R}^{10}$ defined by $T(x) = Ax$. Since $\ker T = \{0\}$, $T$ is injective, so $\dim W \leq \dim \text{null } A$. Thus,

$$\dim \text{null } A^2 \leq \dim \text{null } A + \dim W \leq \dim \text{null } A + \dim \text{null } A = 4 + 4 = 8.$$

It follows that

$$\text{rank } A^2 = 10 - \dim \text{null } A^2 \geq 10 - 8 = 2.$$

To see that this bound can be attained, let $A = \begin{bmatrix} 0 & I_6 \\ 0 & 0 \end{bmatrix}$

where $I_k$ denotes the $k \times k$ identity matrix. Then $A^2 = \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}$, which has rank 2.

(b) Note that $\text{Col } (AB^T) \subset \text{Col } A$, so $\text{rank } AB^T \leq \text{rank } A$. Also, $\text{null } B^T \subset \text{null } (AB^T)$, so $\text{rank } (AB^T) = 10 - \dim \text{null } (AB^T) \leq 10 - \dim \text{null } (B^T) = \text{rank } B^T = \text{rank } B$.

Thus,

$$\text{rank } (AB^T) \leq \min \{ \text{rank } A, \text{rank } B \} = 4.$$

To see that this bound can be attained, let $A = \begin{bmatrix} I_6 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $AB^T = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, which has a rank of 4.
(c) Since the columns of $A$ are orthogonal to the columns of $B$, $\text{Col } A \cap \text{Col } B = \{0\}$. Thus,

$$\text{rank } (A + B) = \dim \text{Col } (A + B) = \dim \text{Col } A + \dim \text{Col } B - \dim (\text{Col } A \cap \text{Col } B) = 6 + 4 - 0 = 10.$$

**Problem 2.** Let $\mathcal{P}^n$ denote the real vector space of polynomials of degree strictly less than $n$. For two functions $f$ and $g$ in $\mathcal{P}^n$, define the inner product by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

(a) Verify that this is an inner product.

(b) Apply the Gram-Schmidt procedure to the basis $\{1, t, t^2\}$ to find an orthogonal basis for $\mathcal{P}^3$.

**Solution**

(a) We show that $\langle \cdot, \cdot \rangle$ satisfies the properties of inner products:

- (symmetry): For all $f, g \in \mathcal{P}^n$, $\langle f, g \rangle = \int_0^1 f(t)g(t)dt = \int_0^1 g(t)f(t)dt = \langle g, f \rangle$.

- (additivity): For all $f, g, h \in \mathcal{P}^n$ and $\alpha, \beta \in \mathbb{R}$,

$$\langle \alpha f + \beta g, h \rangle = \int_0^1 (\alpha f(t) + \beta g(t))h(t)dt$$

$$= \alpha \int_0^1 f(t)h(t)dt + \beta \int_0^1 g(t)h(t)dt$$

$$= \alpha \langle f, h \rangle + \beta \langle g, h \rangle.$$

- (positivity): For $f \neq 0 \in \mathcal{P}^n$, $\langle f, f \rangle = \int_0^1 f(t)^2dt > 0$, and $\langle 0, 0 \rangle = \int_0^1 0dt = 0$. 


(b) Define the orthogonal basis \( \{f_1, f_2, f_3\} \) as follows

\[
f_1(t) = 1
\]
\[
f_2(t) = t - \frac{\langle t, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 = t - \frac{\int_0^1 s \, ds}{\int_0^1 ds} = t - \frac{1}{2}
\]
\[
f_3(t) = t^2 - \frac{\langle t^2, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 - \frac{\langle t^2, f_2 \rangle}{\langle f_2, f_2 \rangle} f_2
\]
\[
= t^2 - \int_0^1 s^2 \, ds - \left( \frac{\int_0^1 (s^3 - \frac{1}{2} s^2) \, ds}{\int_0^1 (s - \frac{1}{2})^2 \, ds} \right) \left( t - \frac{1}{2} \right)
\]
\[
= t^2 - \frac{1}{3} - \frac{1}{12} \left( t - \frac{1}{2} \right)
\]
\[
= t^2 - t + \frac{1}{6}
\]

Problem 3. Suppose \( V \) is a finite-dimensional vector space over \( \mathbb{F} \).

(a) Prove or disprove: if \( S \) and \( T \) are nilpotent operators on \( V \), then \( S + T \) is nilpotent.

(b) Prove or disprove: if \( S \) and \( T \) are nilpotent operators on \( V \) and \( ST = TS \), then \( S + T \) is nilpotent.

(c) Prove if \( S \) is a nilpotent operator on \( V \), then \( I + S \) and \( I - S \) are invertible, where \( I \) is the identity operator on \( V \).

(d) Let \( N \) be an operator on an \( n \)-dimensional vector space, \( n \geq 2 \), such that \( N^n = 0 \), \( N^{n-1} \neq 0 \). Prove there is no operator \( T \) with \( T^2 = N \).

Solution:

(a) The conclusion does not hold. Take

\[
S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\]

as two operators (matrix transformations) on \( \mathbb{R}^2 \). Note

\[
S^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
\]
So $S$ and $T$ are nilpotent. However,

$$S + T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (S + T)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So $S + T$ is not nilpotent.

(b) Let $S^k = 0$ and $T^k = 0$. Then due to commutativity,

$$(S + T)^{k + k} = \sum_{i=0}^{k + k} \binom{k + k}{i} S^i T^{k + k - i}$$

Note if $i < k$, $T^{k + k - i} = 0$; otherwise, $S^i = 0$.

(c) Since $S$ is nilpotent, neither $\pm$ are eigenvalues of $S$. So $\text{null}(I + S) = 0$. Otherwise, $\pm$ are eigenvalues of $S$. So $I + S$ are invertible.

(d) Suppose such $T$ exists. Then $T$ is nilpotent and $T^n = 0$. However, this is a contradiction, since $T^{2n - 2} \neq 0$ and $2n - 2 > n$. 
Problem 4.

$A$ is a real $3 \times 3$ matrix, and we know that

\[
A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}, \quad A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}.
\]

(a) What are the eigenvalues and associated eigenvectors of $A$? Can we use the set of eigenvectors as a basis for $\mathbb{R}^3$? Why or why not? If yes, does this basis have any special properties?

(b) Calculate

\[
A^{2020} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.
\]

(c) Does the linear system $Ax = b$ have a solution for any $b \in \mathbb{R}^3$? If so, why? If not, for what kind of $b \in \mathbb{R}^3$ is $Ax = b$ solvable?

(d) Determine whether matrix $A$ has the following properties. Explain your reasoning.

(i) diagonalizable
(ii) invertible
(iii) orthogonal
(iv) symmetric

Solution

(a) We have the following eigenpairs:

\[
\lambda_1 = -3, \, v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 0, \, v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \lambda_3 = 2, \, v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.
\]

Eigenvectors associated with distinct eigenvalues are not only linearly independent, but also orthogonal. Since any three linearly independent vectors in $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$, $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$. Moreover, since the vectors are mutually orthogonal, they form an orthogonal basis for $\mathbb{R}^3$.

(b) Observe that $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = v_1 + v_2$. Thus,

\[
A^{2020} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = A^{2020}(v_1 + v_2) = \lambda_1^{2020}v_1 + \lambda_2^{2020}v_2 = (-3)^{2020}v_1 + 3^{2020} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]
(c) No. Since $A$ has 0 as an eigenvalue, it is not invertible; so, $\text{Range}(A)$ does not span $\mathbb{R}^3$. Thus, there exists some $b \in \mathbb{R}^3$ for which the linear system $Ax = b$ has no solution.

For $Ax = b$ to have a solution, $b$ must be in the $\text{Range}(A) = \text{span}\{v_1, v_3\}$.

(d) (i) Yes. Since $A$ has distinct eigenvalues, it is diagonalizable.

(ii) No. By part (c), $A$ is not invertible.

(iii) No. For a matrix to be orthogonal, all eigenvalues need to be of modulus 1. The eigenvalues of $A$ are not of modulus 1, so $A$ is not orthogonal.

(iv) Yes. Since $A$ has a basis of orthogonal eigenvectors with real eigenvalues, it is symmetric.
Problem 5.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$T(z_1, z_2, \ldots, z_n) = (z_2 - z_1, z_3 - z_2, \ldots, z_1 - z_n).$$

(a) Give an explicit expression for the adjoint, $T^*$. 

(b) Is $T$ invertible? Explain. 

(c) Find the eigenvalues of $T$. 

Solution

(a) Note that

$$\langle Tu, v \rangle = (u_2 - u_1)v_1 + (u_3 - u_2)v_2 + \cdots + (u_1 - u_n)v_n$$

$$= u_1(v_n - v_1) + u_2(v_1 - v_2) + \cdots + u_n(v_{n-1} - v_n)$$

$$= \langle u, T^*v \rangle.$$ 

Thus, $T^*v = (v_n - v_1, v_1 - v_2, \ldots, v_{n-1} - v_n)$.

(b) Notice that if $v = (c, c, \ldots, c)$ is any constant vector, then $Tv = 0$. Thus, $T$ has a nontrivial null space and is not invertible.

(c) The eigenvalues satisfy $Tv = \lambda v$. Writing this relation in terms of components gives

$$u_2 - u_1 = \lambda u_1 \quad \text{or} \quad u_2 = (1 + \lambda)u_1.$$ 

In general, $u_{j+1} = (1 + \lambda)u_j$, $j = 1, \ldots, n - 1$, and $u_1 = (1 + \lambda)u_n$. Thus,

$$u_1 = (1 + \lambda)u_n = (1 + \lambda)^2u_{n-1} = \cdots = (1 + \lambda)^n u_1.$$ 

This implies that $(1 + \lambda)^n = 1$. Thus, the eigenvalues have the form $\lambda = \mu - 1$, where $\mu$ is any of the $n$th roots of unity, $e^{i2k\pi/n}$, for $k = 0, 1, \ldots, n - 1$. 

Problem 6.
(a) Let $n \geq 2$ and let $V$ be an $n$-dimensional vector space over $\mathbb{C}$ with a set of basis vectors $e_1, \ldots, e_n$. Let $T$ be the linear map of $V$ satisfying

$$T(e_i) = e_{i+1}, \quad i = 1, \ldots, n-1 \quad \text{and} \quad T(e_n) = e_1$$

Is $T$ diagonalizable?

(b) Let $V$ be a finite-dimensional vector space and $T : V \to V$ a diagonalizable linear transformation. Let $W \subseteq V$ be a subspace which is mapped into itself by $T$. Show that the restriction of $T$ to $W$ is diagonalizable.

**Solution:**

(a) The matrix of $T$ with respect to the basis $B = \{e_1, \ldots, e_n\}$ is:

$$M(T, B) = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & & \\
0 & 1 & \cdots & & \\
& & \ddots & \ddots & \\
& & & \ddots & \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}$$

The characteristic polynomial of the matrix is:

$$\det(M(T, B) - \lambda I) = \pm(\lambda^n - 1)$$

which has $n$ distinct roots in $\mathbb{C}$ for all $n$. So the eigenvalues of $T$ are distinct and hence $T$ is diagonalizable.

(b) Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of $T$. Since $T$ is diagonalizable, $V$ can be decomposed as the direct sum of the eigenspaces. So for any $w \in W$, we can UNIQUELY write

$$w = v_1 + \cdots + v_m$$

where each $v_i \in V$ is an eigenvector of $T$ corresponding to eigenvalue $\lambda_i$. Then for any $i \in \{1, 2, \ldots, n\}$, we have

$$\left(\prod_{j \neq i} (T - \lambda_j)\right) w = \left(\prod_{j \neq i} (\lambda_i - \lambda_j)\right) v_i$$

Since $W$ is invariant under $T$, the left hand side in the equation above lies in $W$, and so does the right hand side. So $v_i \in W$ for all $i$. This means $\lambda_1, \ldots, \lambda_m$ are eigenvalues of $T|_W$, and $v_i \in E(\lambda_i, T|_W)$. Based on Eqn (1), $T|_W$ is diagonalizable.
Problem 7. Let $V, W$ be finite-dimensional inner product spaces over $\mathbb{C}$ such that $\dim V \leq \dim W$. Prove that there is a linear map $T : V \to W$ satisfying

$$\langle T(u), T(v) \rangle_W = \langle u, v \rangle_V$$

for all $u, v \in V$.

Solution: Let $v_1, \ldots, v_n$ be an orthonormal basis for $V$ and $w_1, \ldots, w_n, w_{n+1}, \ldots, w_{n+k}$, $k \geq 0$, be an orthonormal basis for $W$. Define $T : V \to W$ such that

$$Tv_j = w_j, \quad j = 1, \ldots, n$$

Such $T$ exists and is unique. For any $u, v \in V$, there are $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{F}$ such that

$$u = \alpha_1 v_1 + \cdots + \alpha_n v_n, \quad v = \beta_1 v_1 + \cdots + \beta_n v_n$$

Then

$$\langle T(u), T(v) \rangle_W = \langle \alpha_1 w_1 + \cdots + \alpha_n w_n, \beta_1 w_1 + \cdots + \beta_n w_n \rangle_W$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle w_i, w_j \rangle_W = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$$

by orthogonality. Also

$$\langle u, v \rangle_V = \langle \alpha_1 v_1 + \cdots + \alpha_n v_n, \beta_1 v_1 + \cdots + \beta_n v_n \rangle_V$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\beta_j} \langle v_i, v_j \rangle_V = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$$

So

$$\langle T(u), T(v) \rangle_W = \langle u, v \rangle_V$$

for all $u, v \in V$. 

---
Problem 8.

Let $V$ be a real finite dimensional inner product space and let $T : V \to V$ be a linear transformation. Assume that $\langle T v, w \rangle = \langle v, T w \rangle$ for all $v, w \in V$.

(a) Prove that if $\lambda$ and $\mu$ are distinct eigenvalues of $T$ then the corresponding eigenspaces $V_\lambda$ and $V_\mu$ are orthogonal.

(b) If $W$ is a subspace of $V$, prove that $T(W) \subseteq W$ implies that $T(W^\perp) \subseteq W^\perp$.

(c) Prove that there exists an eigenvector $v_1 \in V$ for $T$ in $V$ with associated (real) eigenvalue $\lambda_1$. Do not use a big theorem; prove directly. You may assume the fundamental theorem of algebra however.

(d) Prove that there exists an orthonormal basis of $V$ consisting of eigenvectors for $T$.

Solution

We note that the operator $T$ is self-adjoint and we are asked to prove the Spectral Theorem.

(a) Let $\lambda$ and $\mu$ be distinct eigenvalues of $T$. Let $x$ be an eigenvector associated with $\lambda$ and $y$ be an eigenvector associated with $\mu$. We want to prove that $\langle x, y \rangle = 0$.

We have

$$Tx = \lambda x \quad \text{and} \quad Ty = \mu y$$

We consider $\langle Tx, y \rangle$. On the one hand:

$$\langle Tx, y \rangle = \langle \lambda x, y \rangle = \lambda \langle x, y \rangle.$$ 

On the other hand:

$$\langle Tx, y \rangle = \langle x, Ty \rangle = \langle x, \mu y \rangle = \mu \langle x, y \rangle.$$ 

So we have

$$\lambda \langle x, y \rangle = \mu \langle x, y \rangle.$$ 

So we have

$$\begin{align*}
(\lambda - \mu) \langle x, y \rangle &= 0.
\end{align*}$$

We have assumed $\lambda$ and $\mu$ to be distinct eigenvalues, so $\lambda - \mu \neq 0$, so

$$\langle x, y \rangle = 0.$$ 

The eigenspaces $V_\lambda$ and $V_\mu$ are orthogonal.
(b) Let $W$ be a subspace of $V$. We assume that $T(W) \subseteq W$. (In other words, $W$ is invariant under $T$.) We want to show that $T(W^\perp) \subseteq W^\perp$. (In other words, $W^\perp$ is invariant under $T$.)

Let $x \in T(W^\perp)$, we want to show that $x \in W^\perp$.

Let $y \in W$, we want to show that $\langle x, y \rangle = 0$.

Since $x \in T(W^\perp)$, there exists $z \in W^\perp$ such that $x = Tz$.

We have

$$\langle x, y \rangle = \langle Tz, y \rangle = \langle z, Ty \rangle.$$ 

Now we note that $y \in W$, and that $W$ is invariant under $T$, so that $Ty \in W$. Also $z \in W^\perp$, so we get that $\langle z, Ty \rangle = 0$, which proves that $\langle x, y \rangle = 0$.

This proves that

$$(T(W) \subseteq W) \Rightarrow \left(T(W^\perp) \subseteq W^\perp\right).$$

(c) $V$ is finite dimensional. Let $n$ be the dimension of $V$.

We consider an arbitrary nonzero vector $x \in V$. We now consider the set of $n + 1$ vectors:

$$x, \quad Tx, \quad T^2x, \quad T^3x, \quad \ldots \quad T^n x.$$ 

This set consists of $n + 1$ vectors, therefore it is linearly dependent, therefore there exists $n + 1$ not all zeros scalars $a_0, a_1, \ldots, a_n$ such that

$$a_0x + a_1Tx + a_2T^2x + a_3T^3x + \ldots + a_nT^n x = 0. \quad (2)$$

Let $k$ the largest integer such that $a_k \neq 0$. We note that $k$ cannot be zero. $k$ is at least 1.

We now consider the polynomial of degree $k$

$$p(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots + a_n\zeta^n.$$ 

Equation (2) writes:

$$p(T)x = 0.$$ 

Although our problem is in the real setting, we now use complex arithmetic because this makes things easier. By the Fundamental Theorem of Algebra, the polynomial $p$ has $k$ roots (in complex arithmetic). We call them $\zeta_1, \zeta_2, \ldots, \zeta_k$ and we have

$$p(\zeta) = (\zeta - \zeta_1)(\zeta - \zeta_2)\ldots(\zeta - \zeta_k).$$ 

(We note that $k$ is at least 1. So there exists at least one root.)
So we have

\[(T - \zeta_1 I)(T - \zeta_2 I) \cdots (T - \zeta_k I)x = 0.\]

(We note that all these monomials commute. So the order of these monomials is arbitrary. (As is the order of the \(\zeta_i\).)

Now we consider the following \(k\) exclusive cases:

- either \((T - \zeta_k I)x = 0\) and \(x \neq 0\). In this case, \(x\) is an eigenvector of \(T\) of eigenvalue \(\zeta_k\).
- xor \((T - \zeta_{k-1} I)(T - \zeta_k I)x = 0\) and \((T - \zeta_k I)x \neq 0\). In this case, \((T - \zeta_k I)x\) is an eigenvector of \(T\) of eigenvalue \(\zeta_{k-1}\).
- xor \((T - \zeta_{k-2} I)(T - \zeta_{k-1} I)(T - \zeta_k I)x = 0\) and \((T - \zeta_{k-1} I)(T - \zeta_k I)x \neq 0\). In this case, \((T - \zeta_{k-1} I)(T - \zeta_k I)x\) is an eigenvector of \(T\) of eigenvalue \(\zeta_{k-1}\).
- \(\ldots\)
- xor \((T - \zeta_1 I) \cdots (T - \zeta_k I)x = 0\) and \((T - \zeta_2 I) \cdots (T - \zeta_k I)x = 0\). In this case, \((T - \zeta_2 I) \cdots (T - \zeta_k I)x\) is an eigenvector of \(T\) of eigenvalue \(\zeta_1\).

For all of these cases, we find that \(T\) has an eigenvector \(v_1\) and an eigenvalue \(\lambda_1\). As of now, it is possible for \(\lambda_1\) to be complex. It is also possible for \(v_1\) to be complex. \((v_1\) is of the form \((T - \zeta_i I)(T - \zeta_{i-1} I)(T - \zeta_k I)x\) where some \(\zeta_i\) might be complex.) However we now prove that \(\lambda_1\) must be real. And so must \(v_1\) be. Since we have complex vectors and scalars, we must consider the complex inner product \(\langle x, y \rangle_C\) associated with \(\langle x, y \rangle\).

We consider \(\langle v_1, Tv_1 \rangle_C\). One the one hand, we have

\[\langle v_1, Tv_1 \rangle_C = \langle v_1, \lambda_1 v_1 \rangle_C = \lambda_1 \langle v_1, v_1 \rangle_C.\]

One the other hand, we have

\[\langle v_1, Tv_1 \rangle_C = \langle Tv_1, v_1 \rangle_C = \langle \lambda_1 v_1, v_1 \rangle_C = \overline{\lambda_1} \langle v_1, v_1 \rangle_C.\]

We get

\[\lambda_1 \langle v_1, v_1 \rangle_C = \overline{\lambda_1} \langle v_1, v_1 \rangle_C.\]

And, since \(v_1 \neq 0\), \(\langle v_1, v_1 \rangle_C \neq 0\), so

\[\lambda_1 = \overline{\lambda_1}.\]

So

\[\lambda_1 \in \mathbb{R}.\]

(And then so is \(v_1\).)
(d) We start by using part (c) to find an eigenvector $v_1$ and an associated (real) eigenvalue $\lambda_1$ of $T$.

We call $W_1 = \text{Span}(v_1)$. $W_1$ is invariant under $T$, so, by using part (b), we see that $W_1^\perp$ is invariant under $T$. Therefore we can consider $T_1$, the restriction of $T$ to $W_1^\perp$. It is important to observe that, due to the invariance of $W_1^\perp$, $T_1$ is an operator. We have: $T_1 : W_1^\perp \mapsto W_1^\perp$. Since $T_1$ is an operator, it makes sense to speak about eigenvalues and eigenvectors for $T_1$. Also, any eigenvalues and eigenvectors of $T_1$ will also be eigenvalues and eigenvectors of $T$. It is also important to note that $T_1$ is self-adjoint. All this to say that, we can use part (c) on $T_1$ to find an eigenvector $v_2 \in W_1^\perp$ and an associated (real) eigenvalue $\lambda_2$ of $T_1$. This eigencouple $(v_2, \lambda_2)$ of $T_1$ is also an eigencouple of $T$, and we have that $v_1$ and $v_2$ are mutually orthogonal eigenvectors of $T$.

We call $W_2 = \text{Span}(v_1, v_2)$. We note that $v_1$ and $v_2$ are two eigenvectors of $T$ and there are mutually orthogonal. $W_2$ is invariant under $T$, (as any subspace spanned by eigenvectors of $T$ is,) so, by using part (b), we see that $W_2^\perp$ is invariant under $T$. Therefore we can consider $T_2$, the restriction of $T$ to $W_2^\perp$. It is important to observe that, due to the invariance of $W_2^\perp$, $T_2$ is an operator. We have: $T_2 : W_2^\perp \mapsto W_2^\perp$. Since $T_2$ is an operator, it makes sense to speak about eigenvalues and eigenvectors for $T_2$. Also, any eigenvalues and eigenvectors of $T_2$ will also be eigenvalues and eigenvectors of $T$. It is also important to note that $T_2$ is self-adjoint. All this to say that, we can use part (c) on $T_2$ to find an eigenvector $v_3 \in W_2^\perp$ and an associated (real) eigenvalue $\lambda_3$ of $T_2$. This eigencouple $(v_3, \lambda_3)$ of $T_2$ is also an eigencouple of $T$, and we have $v_1$, $v_2$, and $v_3$ are mutually orthogonal eigenvectors of $T$.

We can continue in this manner until we find a basis of $V$ made of $n$ mutually orthogonal eigenvectors of $A$.

The last step is to normalize our $n$ vectors so as to obtain an orthonormal basis of $V$ made of eigenvectors of $T$. 