University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 24, 2020

Name: ____

Exam Rules:

- This is a closed book exam. Take your time to read each problem carefully. Once the exam begin, you have 4 hours to complete the exam.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do no submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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Part I. Work **all** of problems 1 through 4.

Problem 1. Let A and B be two real 10×10 matrices. Suppose that the rank of A is 6 and the rank of B is 4. Justify your answers to the following questions.

(a) What is the minimum possible rank of the matrix A^2

(b) What is the maximum possible rank of the matrix AB^T?

(c) If the columns of A are orthogonal to the columns of B, must the rank of $A + B$ be equal to 10?

Problem 2. Let $\mathcal{P}^n$ denote the real vector space of polynomials of degree strictly less than n. For two functions f and g in $\mathcal{P}^n$, define the inner product by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

(a) Verify that this is an inner product.

(b) Apply the Gram-Schmidt procedure to the basis $\{1, t, t^2\}$ to find an orthogonal basis for $\mathcal{P}^3$.

Problem 3. Suppose V is a finite-dimensional vector space over $\mathbb{F}$.

(a) Prove or disprove: if S and T are nilpotent operators on V, then $S+T$ is nilpotent.

(b) Prove or disprove: if S and T are nilpotent operators on V and $ST = TS$, then $S + T$ is nilpotent.

(c) Prove if S is a nilpotent operator on V, then $I + S$ and $I - S$ are invertible, where I is the identity operator on V.

(d) Let N be an operator on an n-dimensional vector space, $n \geq 2$, such that $N^n = 0$, $N^{n-1} \neq 0$. Prove there is no operator T with $T^2 = N$.

Problem 4.

A is a real 3×3 matrix, and we know that

$$A\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}-3\\-3\\-3\end{pmatrix},\quad A\begin{pmatrix}-1\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix},\quad A\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}2\\2\\-4\end{pmatrix}.$$

(a) What are the eigenvalues and associated eigenvectors of A? Can we use the set of eigenvectors as a basis for $\mathbb{R}^3$? Why or why not? If yes, does this basis have any special properties?

(b) Calculate

$$A^{2020}\begin{pmatrix}0\\2\\1\end{pmatrix}.$$

(c) Does the linear system $Ax = b$ have a solution for any $b \in \mathbb{R}^3$? If so, why? If not, for what kind of $b \in \mathbb{R}^3$ is $Ax = b$ solvable?

(d) Determine whether matrix A has the following properties. Explain your reasoning.

(i) diagonalizable

(ii) invertible

(iii) orthogonal

(iv) symmetric

Part II. Work **two** of problems 5 through 8.

Problem 5.

We consider the inner product space $\mathbb{R}^n$ with its standard inner product. ($\langle u, v\rangle = u_1v_1 + \ldots + u_nv_n$.) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$T(z_1, z_2, \ldots, z_n) = (z_2 - z_1, z_3 - z_2, \ldots, z_1 - z_n).$$

(a) Give an explicit expression for the adjoint, T^*.

(b) Is T invertible? Explain.

(c) Find the eigenvalues of T.

Problem 6.

(a) Let $n \geq 2$ and Let V be an n-dimensional vector space over $\mathbb{C}$ with a set of basis vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. Let T be the linear map of V satisfying

$$T(\boldsymbol{e}_i) = \boldsymbol{e}_{i+1}, i = 1, \ldots, n-1 \quad \text{and } T(\boldsymbol{e}_n) = \boldsymbol{e}_1$$

Is T diagonalizable?

(b) Let V be a finite-dimensional vector space and $T : V \to V$ a diagonalizable linear transformation. Let $W \subseteq V$ be a subspace which is mapped into itself by T. Show that the restriction of T to W is diagonalizable.

Problem 7. Let V, W be finite-dimensional inner product spaces over $\mathbb{C}$ such that $\dim V \leq \dim W$. Prove that there is a linear map $T : V \to W$ satisfying

$$\langle T(\boldsymbol{u}), T(\boldsymbol{v})\rangle_W = \langle \boldsymbol{u}, \boldsymbol{v}\rangle_V$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$.

Problem 8.

Let V be a real finite dimensional inner product space and let $T : V \to V$ be a linear transformation. Assume that $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all v, $w \in V$.

(a) Prove that if λ and μ are distinct eigenvalues of T then the corresponding eigenspaces V_λ and V_μ are orthogonal.

(b) If W is a subspace of V, prove that $T(W) \subseteq W$ implies that $T(W^\perp) \subseteq W^\perp$.

(c) Prove that there exists an eigenvector $v_1 \in V$ for T in V with associated (real) eigenvalue λ_1. Do not use a big theorem; prove directly. You may assume the fundamental theorem of algebra however.

(d) Prove that there exists an orthonormal basis of V consisting of eigenvectors for T.