Name:  

**Exam Rules:**

- This is a closed book exam. Take your time to read each problem carefully. Once the exam begin, you have 4 hours to complete the exam.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do no submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- **Notation:** Throughout the exam, \( \mathbb{R} \) and \( \mathbb{C} \) denote the sets of real and complex numbers, respectively. \( F \) denotes either \( \mathbb{R} \) or \( \mathbb{C} \). \( F^n \) and \( F^{n,n} \) are the vector spaces of \( n \)-tuples and \( n \times n \) matrices, respectively, over the field \( F \). \( \mathcal{L}(V) \) denotes the set of linear operators on the vector space \( V \). \( T^* \) is the adjoint of the operator \( T \) and \( \lambda^* \) is the complex conjugate of the scalar \( \lambda \). In an inner product space \( V \), \( U^\perp \) denotes the orthogonal complement of the subspace \( U \).
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

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DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

**Applied Linear Algebra Preliminary Exam Committee:**
Stephen Billups (Chair), Julien Langou, Yaning Liu
Part I. Work all of problems 1 through 4.

Problem 1. Let $A$ and $B$ be two real $10 \times 10$ matrices. Suppose that the rank of $A$ is 6 and the rank of $B$ is 4. Justify your answers to the following questions.

(a) What is the minimum possible rank of the matrix $A^2$?

(b) What is the maximum possible rank of the matrix $AB^T$?

(c) If the columns of $A$ are orthogonal to the columns of $B$, must the rank of $A + B$ be equal to 10?

Problem 2. Let $\mathcal{P}^n$ denote the real vector space of polynomials of degree strictly less than $n$. For two functions $f$ and $g$ in $\mathcal{P}^n$, define the inner product by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$ 

(a) Verify that this is an inner product.

(b) Apply the Gram-Schmidt procedure to the basis $\{1, t, t^2\}$ to find an orthogonal basis for $\mathcal{P}^3$.

Problem 3. Suppose $V$ is a finite-dimensional vector space over $\mathbb{F}$.

(a) Prove or disprove: if $S$ and $T$ are nilpotent operators on $V$, then $S + T$ is nilpotent.

(b) Prove or disprove: if $S$ and $T$ are nilpotent operators on $V$ and $ST = TS$, then $S + T$ is nilpotent.

(c) Prove if $S$ is a nilpotent operator on $V$, then $I + S$ and $I - S$ are invertible, where $I$ is the identity operator on $V$.

(d) Let $N$ be an operator on an $n$-dimensional vector space, $n \geq 2$, such that $N^n = 0$, $N^{n-1} \neq 0$. Prove there is no operator $T$ with $T^2 = N$. 
Problem 4.

A is a real $3 \times 3$ matrix, and we know that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}, \quad A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}.$$  

(a) What are the eigenvalues and associated eigenvectors of $A$? Can we use the set of eigenvectors as a basis for $\mathbb{R}^3$? Why or why not? If yes, does this basis have any special properties?

(b) Calculate

$$A^{2020} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$  

(c) Does the linear system $Ax = b$ have a solution for any $b \in \mathbb{R}^3$? If so, why? If not, for what kind of $b \in \mathbb{R}^3$ is $Ax = b$ solvable?

(d) Determine whether matrix $A$ has the following properties. Explain your reasoning.

(i) diagonalizable

(ii) invertible

(iii) orthogonal

(iv) symmetric
Part II. Work **two** of problems 5 through 8.

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**Problem 5.**

We consider the inner product space $\mathbb{R}^n$ with its standard inner product. \( \langle u, v \rangle = u_1v_1 + \ldots + u_nv_n \) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

\[
T(z_1, z_2, \ldots, z_n) = (z_2 - z_1, z_3 - z_2, \ldots, z_1 - z_n).
\]

(a) Give an explicit expression for the adjoint, $T^*$.  

(b) Is $T$ invertible? Explain. 

(c) Find the eigenvalues of $T$.  

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**Problem 6.**  

(a) Let $n \geq 2$ and let $V$ be an $n$-dimensional vector space over $\mathbb{C}$ with a set of basis vectors $e_1, \ldots, e_n$. Let $T$ be the linear map of $V$ satisfying

\[
T(e_i) = e_{i+1}, \quad i = 1, \ldots, n-1 \quad \text{and} \quad T(e_n) = e_1.
\]

Is $T$ diagonalizable? 

(b) Let $V$ be a finite-dimensional vector space and $T : V \to V$ a diagonalizable linear transformation. Let $W \subseteq V$ be a subspace which is mapped into itself by $T$. Show that the restriction of $T$ to $W$ is diagonalizable.  

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**Problem 7.**  

Let $V$, $W$ be finite-dimensional inner product spaces over $\mathbb{C}$ such that $\dim V \leq \dim W$. Prove that there is a linear map $T : V \to W$ satisfying

\[
\langle T(u), T(v) \rangle_W = \langle u, v \rangle_V
\]

for all $u, v \in V$.  

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**Problem 8.**  

Let $V$ be a real finite dimensional inner product space and let $T : V \to V$ be a linear transformation. Assume that $\langle T v, w \rangle = \langle v, T w \rangle$ for all $v, w \in V$.  

(a) Prove that if \( \lambda \) and \( \mu \) are distinct eigenvalues of \( T \) then the corresponding eigenspaces \( V_\lambda \) and \( V_\mu \) are orthogonal.

(b) If \( W \) is a subspace of \( V \), prove that \( T(W) \subseteq W \) implies that \( T(W^\perp) \subseteq W^\perp \).

(c) Prove that there exists an eigenvector \( v_1 \in V \) for \( T \) in \( V \) with associated (real) eigenvalue \( \lambda_1 \). Do not use a big theorem; prove directly. You may assume the fundamental theorem of algebra however.

(d) Prove that there exists an orthonormal basis of \( V \) consisting of eigenvectors for \( T \).