University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 24, 2020

Name: ____

Exam Rules:

- This is a closed book exam. Take your time to read each problem carefully. Once the exam begin, you have 4 hours to complete the exam.
- There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part. If you do, only the first two attempted problems will be graded. Each problem is worth 20 points.
- Do no submit multiple alternative solutions to any problem; if you do, only the first solution will be graded.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- If you are confused or stuck on a problem, either ask a question or move on to another problem.

Problem	Points	Score		Problem	Points	Score
1.	20			5.	20	
2.	20			6.	20	
3.	20			7.	20	
4.	20			8.	20	
				Total	120	

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Stephen Billups (Chair), Julien Langou, Yaning Liu

Part I. Work **all** of problems 1 through 4.

Problem 1. Let A and B be two real 10×10 matrices. Suppose that the rank of A is 6 and the rank of B is 4. Justify your answers to the following questions.

(a) What is the minimum possible rank of the matrix A^2

(b) What is the maximum possible rank of the matrix AB^T?

(c) If the columns of A are orthogonal to the columns of B, must the rank of $A+B$ be equal to 10?

Solution

(a) Note that null $A^2 =$ null $A + W$, where $W = \{x \in \text{Row } A | Ax \in \text{null } A\}$. Consider the mapping $T : \text{Row } A \to \mathbb{R}^{10}$ defined by $T(x) = Ax$. Since $\ker T = \{0\}$, T is injective, so $\dim W \leq \dim \text{null } A$. Thus,

$$\dim \text{null } A^2 \leq \dim \text{null } A + \dim W \leq \dim \text{null } A + \dim \text{null } A = 4 + 4 = 8.$$

It follows that

$$\text{rank } A^2 = 10 - \dim \text{null } A^2 \geq 10 - 8 = 2.$$

To see that this bound can be attained, let

$$A = \begin{bmatrix} 0 & I_6 \\ 0 & 0 \end{bmatrix}$$

where I_k denotes the $k \times k$ identity matrix. Then $A^2 = \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}$, which has rank 2.

(b) Note that $\text{Col}\,(AB^T) \subset \text{Col } A$, so $\text{rank } AB^T \leq \text{rank } A$. Also, $\text{null } B^T \subset \text{null}\,(AB^T)$, so $\text{rank}\,(AB^T) = 10 - \dim \text{null}\,(AB^T) \leq 10 - \dim \text{null}\,(B^T) = \text{rank } B^T = \text{rank } B$. Thus,

$$\text{rank}\,(AB^T) \leq \min\{\text{rank } A, \text{rank } B\} = 4.$$

To see that this bound can be attained, let $A = \begin{bmatrix} I_6 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB^T = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$, which has a rank of 4.

(c) This is false. The idea of the following counterexample is that in general, $\mathrm{col}(A + B) \neq \mathrm{col}(A) + \mathrm{col}(B)$:

$$A = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0 \end{bmatrix}, \quad B = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0 \end{bmatrix}$$

Problem 2. Let $\mathcal{P}^n$ denote the real vector space of polynomials of degree strictly less than n. For two functions f and g in $\mathcal{P}^n$, define the inner product by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

(a) Verify that this is an inner product.

(b) Apply the Gram-Schmidt procedure to the basis $\{1, t, t^2\}$ to find an orthogonal basis for $\mathcal{P}^3$.

Solution

(a) We show that $\langle \cdot, \cdot \rangle$ satisfies the properties of inner products:

- (symmetry): For all $f, g \in \mathcal{P}^n$, $\langle f, g \rangle = \int_0^1 f(t)g(t)dt = \int_0^1 g(t)f(t)dt = \langle g, f \rangle$.

- (additivity): For all $f, g, h \in \mathcal{P}^n$ and $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}\langle \alpha f + \beta g, h \rangle &= \int_0^1 (\alpha f(t) + \beta g(t)) h(t) dt \\ &= \alpha \int_0^1 f(t)h(t)dt + \beta \int_0^1 g(t)h(t)dt \\ &= \alpha \langle f, h \rangle + \beta \langle g, h \rangle .\end{aligned}$$

- (positivity): For $f \neq 0 \in \mathcal{P}^n$, $\langle f, f \rangle = \int_0^1 f(t)^2 d > 0$, and $\langle 0, 0 \rangle = \int_0^1 0dt = 0$.

(b) Define the orthogonal basis $\{f_1, f_2, f_3\}$ as follows

$$\begin{aligned}f_1(t) &= 1 \\ f_2(t) &= t - \frac{\langle t, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 = t - \frac{\int_0^1 sds}{\int_0^1 ds} = t - \frac{1}{2} \\ f_3(t) &= t^2 - \frac{\langle t^2, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1 - \frac{\langle t_2, f_2 \rangle}{\langle f_2, f_2 \rangle} f_2 \\ &= t^2 - \int_0^1 s^2 ds - \left(\frac{\int_0^1 \left(s^3 - \frac{1}{2}s^2\right) ds}{\int_0^1 (s - \frac{1}{2})^2 ds)} \right) \left(t - \frac{1}{2} \right) \\ &= t^2 - \frac{1}{3} - \frac{1/12}{1/12} \left(t - \frac{1}{2} \right) \\ &= t^2 - t + \frac{1}{6}\end{aligned}$$

Problem 3. Suppose V is a finite-dimensional vector space over $\mathbb{F}$.

(a) Prove or disprove: if S and T are nilpotent operators on V, then $S+T$ is nilpotent.

(b) Prove or disprove: if S and T are nilpotent operators on V and $ST = TS$, then $S + T$ is nilpotent.

(c) Prove if S is a nilpotent operator on V, then $I + S$ and $I - S$ are invertible, where I is the identity operator on V.

(d) Let N be an operator on an n-dimensional vector space, $n \geq 2$, such that $N^n = 0$, $N^{n-1} \neq 0$. Prove there is no operator T with $T^2 = N$.

Solution:

(a) The conclusion does not hold. Take

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

as two operators (matrix transformations) on $\mathbb{F}^2$. Note

$$S^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So S and T are nilpotent. However,

$$S + T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, (S+T)^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So $S + T$ is not nilpotent.

(b) Let $S^{k_S} = 0$ and $T^{k_T} = 0$. Then due to commutativity,

$$(S+T)^{k_S+k_T} = \sum_{i=0}^{k_S+k_T} \binom{k_S + k_T}{i} S^i T^{k_S+k_T-i}$$

Note if $i < k_S$, $T^{k_S+k_T-i} = 0$; otherwise, $S^i = 0$.

(c) Since S is nilpotent, neither ± 1 are eigenvalues of S. So $\operatorname{null}(I \pm S) = \mathbf{0}$. Otherwise, ± 1 are eigenvalues of S. So $I \pm S$ are invertible.

(d) Suppose such T exists. Then T is nilpotent and $T^n = 0$. However, this is a contradiction, since $T^{2n-2} \neq 0$ and $2n - 2 > n$. (We proved that N does not have a square root.)

Problem 4.

A is a real 3×3 matrix, and we know that

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}, \quad A \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad A \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}.$$

(a) What are the eigenvalues and associated eigenvectors of A? Can we use the set of eigenvectors as a basis for $\mathbb{R}^3$? Why or why not? If yes, does this basis have any special properties?

(b) Calculate

$$A^{2020} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

(c) Does the linear system $Ax = b$ have a solution for any $b \in \mathbb{R}^3$? If so, why? If not, for what kind of $b \in \mathbb{R}^3$ is $Ax = b$ solvable?

(d) Determine whether matrix A has the following properties. Explain your reasoning.

(i) diagonalizable

(ii) invertible

(iii) orthogonal

(iv) symmetric

---

1. **What are the eigenvalues and associated eigenvectors of A?** We have the following eigencouples:

$$(\lambda_1 = -3, v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) \quad (\lambda_2 = 0, v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}) \quad \text{and} \quad (\lambda_3 = 2, v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}).$$

Can we use the set of eigenvectors as a basis for $\mathbb{R}^3$?

Yes, this set of three eigenvectors (v_1, v_2, v_3) is a basis of $\mathbb{R}^3$.

Why or why not?

Few reasons why "yes".

First, we can see that these three vectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

are linearly independent. Three linearly independent vectors in a space of dimension 3 form a basis.

Second, we see that the three associated eigenvalues

$$\lambda_1 = -3, \quad \lambda_2 = 0, \quad \text{and} \quad \lambda_3 = 2.$$

are distinct. So the three associated eigenvectors are linearly independent. Three linearly independent vectors in a space of dimension 3 form a basis.

If yes, does this basis have any special properties?

We can see that these three vectors are mutually orthogonal. In other worlds: $v_1^T v_2 = 0$, $v_2^T v_3 = 0$, and $v_1^T v_3 = 0$. So (v_1, v_2, v_3) is not only a basis of $\mathbb{R}^3$, it is an **orthogonal** basis of $\mathbb{R}^3$.

Here we should probably realize two quick things.

(a) We can normalize each of these vectors (v_1, v_2, v_3) so as to obtain (q_1, q_2, q_3), an orthonormal basis of $\mathbb{R}^3$ of eigenvectors of A. We get:

$$q_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}) \quad q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad q_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

(b) We should realize that, since (1) A has an orthonormal basis of (real) eigenvectors, and (2) the eigenvalues of A are real, then A is a real **symmetric** matrix.

2. Calulate

$$A^{2020} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Let us call x such that:

$$x = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

We see that

$$x = v_1 + v_2.$$

So

$$Ax = A(v_1 + v_2) = -3v_1; \quad A^2x = 3^2v_1; \quad A^3x = -3^3v_1 \quad \dots \quad A^{2020}x = 3^{2020}v_1.$$

So

$$A^{2020} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = 3^{2020} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

3. **Does the linear system $Ax = b$ have a solution for any $b \in \mathbb{R}^3$?**

No, there exist some $b \in \mathbb{R}^3$ for which the linear system $Ax = b$ has no solution.

If so, why?

Few reasons:

(a) Since $\dim(\text{Null}(A)) = 1$, then $\dim(\text{Range}(A)) = 3 - 1 = 2$. So, since $\dim(\text{Range}(A)) = 2$, $\text{Range}(A)$ does not span $\mathbb{R}^3$, so there exist some $b \in \mathbb{R}^3$ for which the linear system $Ax = b$ has no solution.

(b)

$$\text{Range}(A) = \text{Span}(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}).$$

So for example, there is no solution when b is

$$b = v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad b = v_1 + v_2 + v_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \quad \text{or etc.}$$

(As long as the coefficient on v_2 is not zero, there is no solution.)

If not, for what kind of $b \in \mathbb{R}^3$ is $Ax = b$ solvable?

$Ax = b$ is solvable if and only if $b \in \text{Range}(A)$ if and only if

$$b \in \text{Span}(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

4. **Determine whether matrix A has the following properties. Explain your reasoning.**

   (a) **diagonalizable**
   Certainly yes.

   (b) **invertible**
   Certainly not.

   (c) **orthogonal**
   No. Few reasons again. For a matrix to be orthogonal, all eigenvalues need to be of modulus 1. The eigenvalues of A are certainly not of modulus 1.

   (d) **symmetric**
   Yes! Basis of orthogonal eigenvectors and real eigenvalues implies symmetric.

We note that, while not needed to answer any of the questions asked, we can compute the matrix A explicitly. Either using

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 1 \\ -3 & 0 & 1 \\ -3 & 0 & -2 \end{pmatrix}$$

$$A \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 1 \\ -3 & 0 & 1 \\ -3 & 0 & -2 \end{pmatrix}$$

Therefore

$$A = \begin{pmatrix} -3 & 0 & 1 \\ -3 & 0 & 1 \\ -3 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix}^{-1} = \frac{1}{6} \begin{pmatrix} -3 & 0 & 1 \\ -3 & 0 & 1 \\ -3 & 0 & -2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ -3 & 3 & 0 \\ 1 & 1 & -2 \end{pmatrix} = -\frac{1}{3} \begin{pmatrix} 2 & 2 & 5 \\ 2 & 2 & 5 \\ 5 & 5 & -1 \end{pmatrix}.$$

Or using $A = VDV^{-1}$ with

$$V = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

We get that

$$V^{-1} = \frac{1}{6} \begin{pmatrix} 2 & 2 & 2 \\ -3 & 3 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$

We find

$$A = -\frac{1}{3} \begin{pmatrix} 2 & 2 & 5 \\ 2 & 2 & 5 \\ 5 & 5 & -1 \end{pmatrix}.$$

From this, we can, for example, see that A is symmetric. We can see that $A^T A$ is not identity, so that A is not orthogonal.

Part II. Work **two** of problems 5 through 8.

Problem 5.

We consider the inner product space $\mathbb{R}^n$ with its standard inner product. ($\langle u, v\rangle = u_1v_1 + \ldots + u_nv_n$.) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$T(z_1, z_2, \ldots, z_n) = (z_2 - z_1, z_3 - z_2, \ldots, z_1 - z_n).$$

(a) Give an explicit expression for the adjoint, T^*.

(b) Is T invertible? Explain.

(c) Find the eigenvalues of T.

Solution

(a) Note that

$$\begin{aligned}\langle Tu, v\rangle &= (u_2 - u_1)v_1 + (u_3 - u_2)v_2 + \cdots + (u_1 - u_n)v_n \\ &= u_1(v_n - v_1) + u_2(v_1 - v_2) + \ldots + u_n(v_{n-1} - v_n) \\ &= \langle u, T^*v\rangle .\end{aligned}$$

Thus, $T^*v = (v_n - v_1, v_1 - v_2, \ldots, v_{n-1} - v_n)$.

(b) Notice that if $v = (c, c, \ldots, c)$ is any constant vector, then $Tv = 0$. Thus, T has a nontrivial null space and is not invertible.

(c) The eigenvalues satisfy $Tv = \lambda v$. Writing this relation in terms of components gives

$$u_2 - u_1 = \lambda u_1 \quad \text{or} \quad u_2 = (1 + \lambda)u_1.$$

In general, $u_{j+1} = (1 + \lambda)u_j$, $j = 1, \ldots, n - 1$, and $u_1 = (1 + \lambda)u_n$. Thus,

$$u_1 = (1 + \lambda)u_n = (1 + \lambda)^2 u_{n-1} = \cdots = (1 + \lambda)^n u_1.$$

This implies that $(1 + \lambda)^n = 1$. Thus, the eigenvalues have the form $\lambda = \mu - 1$, where μ is any of the nth roots of unity, $e^{i2k\pi/n}$, for $k = 0, 1, \ldots, n - 1$.

Problem 6.

(a) Let $n \geq 2$ and Let V be an n-dimensional vector space over $\mathbb{C}$ with a set of basis vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. Let T be the linear map of V satisfying

$$T(\boldsymbol{e}_i) = \boldsymbol{e}_{i+1}, i = 1, \ldots, n-1 \quad \text{and } T(\boldsymbol{e}_n) = \boldsymbol{e}_1$$

Is T diagonalizable?

(b) Let V be a finite-dimensional vector space and $T : V \to V$ a diagonalizable linear transformation. Let $W \subseteq V$ be a subspace which is mapped into itself by T. Show that the restriction of T to W is diagonalizable.

Solution:

(a) The matrix of T with respect to the basis $B = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is:

$$\mathcal{M}(T, B) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & & \\ 0 & 1 & 0 & \cdots & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of the matrix is:

$$\det(\mathcal{M}(T, B) - \lambda I) = \pm(\lambda^n - 1)$$

which has n distinct roots in $\mathcal{C}$ for all n. So the eigenvalues of T are distinct and hence T is diagonalizable.

(b) Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Since T is diagonalizable, V can be decomposed as the direct sum of the eigenspaces. So for any $\boldsymbol{w} \in W$, we can UNIQUELY write

$$\boldsymbol{w} = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_m \tag{1}$$

where each $v_i \in V$ is an eigenvector of T corresponding to eigenvalue λ_i. Then for any $i \in \{1, 2, \ldots, n\}$, we have

$$\left(\prod_{j \neq i} (T - \lambda_j) \right) \boldsymbol{w} = \left(\prod_{j \neq i} (\lambda_i - \lambda_j) \right) \boldsymbol{v}_i$$

Since W is invariant under T, the left hand side in the equation above lies in W, and so does the right hand side. So $\boldsymbol{v}_i \in W$ for all i. This means $\lambda_1, \ldots, \lambda_m$ are eigenvalues of $T|_W$, and $v_i \in E(\lambda_i, T|_W)$. Based on Eqn (1), $T|_W$ is diagonalizable.

Problem 7. Let V, W be finite-dimensional inner product spaces over $\mathbb{C}$ such that $\dim V \le \dim W$. Prove that there is a linear map $T : V \to W$ satisfying

$$\langle T(\boldsymbol{u}), T(\boldsymbol{v})\rangle_W = \langle \boldsymbol{u}, \boldsymbol{v}\rangle_V$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$.

Solution: Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ be an orthonormal basis for V and $\boldsymbol{w}_1, \dots, \boldsymbol{w}_n, \boldsymbol{w}_{n+1}, \dots, \boldsymbol{w}_{n+k}$, $k \ge 0$, be an orthonormal basis for W. Define $T : V \to W$ such that

$$T\boldsymbol{v}_j = \boldsymbol{w}_j, \quad j = 1, \dots, n$$

Such T exists and is unique. For any $\boldsymbol{u}, \boldsymbol{v} \in V$, there are $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{F}$ such that

$$\boldsymbol{u} = \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n, \quad \boldsymbol{v} = \beta_1 \boldsymbol{v}_1 + \cdots + \beta_n \boldsymbol{v}_n$$

Then

$$\begin{aligned} \langle T(\boldsymbol{u}), T(\boldsymbol{v})\rangle_W &= \langle \alpha_1 \boldsymbol{w}_1 + \cdots + \alpha_n \boldsymbol{w}_n, \beta_1 \boldsymbol{w}_1 + \cdots + \beta_n \boldsymbol{w}_n\rangle_W \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\beta}_j \langle \boldsymbol{w}_i, \boldsymbol{w}_j\rangle_W = \sum_{i=1}^{n} \alpha_i \overline{\beta}_i \end{aligned}$$

by orthogonality. Also

$$\begin{aligned} \langle \boldsymbol{u}, \boldsymbol{v}\rangle_V &= \langle \alpha_1 \boldsymbol{v}_1 + \cdots + \alpha_n \boldsymbol{v}_n, \beta_1 \boldsymbol{v}_1 + \cdots + \beta_n \boldsymbol{v}_n\rangle_V \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\beta}_j \langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle_V = \sum_{i=1}^{n} \alpha_i \overline{\beta}_i \end{aligned}$$

So

$$\langle T(\boldsymbol{u}), T(\boldsymbol{v})\rangle_W = \langle \boldsymbol{u}, \boldsymbol{v}\rangle_V$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in V$.

Problem 8.

Let V be a real finite dimensional inner product space and let $T : V \to V$ be a linear transformation. Assume that $\langle Tv, w\rangle = \langle v, Tw\rangle$ for all v, $w \in V$.

(a) Prove that if λ and μ are distinct eigenvalues of T then the corresponding eigenspaces V_λ and V_μ are orthogonal.

(b) If W is a subspace of V, prove that $T(W) \subseteq W$ implies that $T(W^{\perp}) \subseteq W^{\perp}$.

(c) Prove that there exists an eigenvector $v_1 \in V$ for T in V with associated (real) eigenvalue λ_1. Do not use a big theorem; prove directly. You may assume the fundamental theorem of algebra however.

(d) Prove that there exists an orthonormal basis of V consisting of eigenvectors for T.

Solution

We note that the operator T is self-adjoint and we are asked to prove the Spectral Theorem.

(a) Let λ and μ be distinct eigenvalues of T. Let x be an eigenvector associated with λ and y be an eigenvector associated with μ. We want to prove that $\langle x, y\rangle = 0$.

We have

$$Tx = \lambda x \quad \text{and} \quad Ty = \mu y$$

We consider $\langle Tx, y\rangle$. On the one hand:

$$\langle Tx, y\rangle = \langle \lambda x, y\rangle = \lambda \langle x, y\rangle .$$

On the other hand:

$$\langle Tx, y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \mu \langle x, y\rangle .$$

So we have

$$\lambda \langle x, y\rangle = \mu \langle x, y\rangle .$$

So we have

$$(\lambda - \mu) \langle x, y\rangle = 0.$$

We have assumed λ and μ to be distinct eigenvalues, so $(\lambda - \mu) \neq 0$, so

$$\langle x, y\rangle = 0.$$

The eigenspaces V_λ and V_μ are orthogonal.

(b) Let W be a subspace of V. We assume that $T(W) \subseteq W$. (In other words, W is invariant under T.) We want to show that $T(W^{\perp}) \subseteq W^{\perp}$. (In other words, $W^{\perp}$ is invariant under T.)

Let $x \in T(W^{\perp})$, we want to show that $x \in W^{\perp}$.

Let $y \in W$, we want to show that $\langle x, y\rangle = 0$.

Since $x \in T(W^{\perp})$, there exists $z \in W^{\perp}$ such that $x = Tz$.

We have

$$\langle x, y \rangle = \langle Tz, y \rangle = \langle z, Ty \rangle.$$

Now we note that $y \in W$, and that W is invariant under T, so that $Ty \in W$. Also $z \in W^\perp$, so we get that $\langle z, Ty \rangle = 0$, which proves that

$$\langle x, y \rangle = 0.$$

This proves that

$$(T(W) \subseteq W) \Rightarrow \left(T(W^\perp) \subseteq W^\perp\right).$$

(c) V is finite dimensional. Let n be the dimension of V.

We consider an arbitrary nonzero vector $x \in V$. We now consider the set of $n+1$ vectors:

$$x, \quad Tx, \quad T^2x, \quad T^3x, \quad \dots \quad T^nx.$$

This set consists of $n+1$ vectors, therefore it is linearly dependent, therefore there exists $n+1$ not all zeros scalars a_0, a_1, ..., a_n such that

$$a_0x + a_1Tx + a_2T^2x + a_3T^3x + \dots + a_nT^nx = 0. \tag{2}$$

Let k the largest integer such that $a_k \neq 0$. We note that k cannot be zero. k is at least 1.

We now consider the polynomial of degree k

$$p(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + a_3\zeta^3 + \dots + a_n\zeta^n.$$

Equation (2) writes:

$$p(T)x = 0.$$

Although our problem is in the real settng, we now use complex arithmetic because this makes thing easier. By the Fundamental Theorem of Algebra, the polynomial p has k roots (in complex arithemtic). We call them ζ_1, ζ_2, ..., ζ_k and we have

$$p(\zeta) = (\zeta - \zeta_1)(\zeta - \zeta_2)\dots(\zeta - \zeta_k).$$

(We note that k is at least 1. So there exists at least one root.)

So we have

$$(T - \zeta_1 I)(T - \zeta_2 I)\dots(T - \zeta_k I)x = 0.$$

(We note that all these monomials commute. So the order of these monomials is arbritrary. (As is the order of the ζ_i.)

Now we consider the following k exclusive cases:

- either $(T - \zeta_k I)x = 0$ and $x \neq 0$. In this case, x is an eigenvector of T of eigenvalue ζ_k.
- xor $(T - \zeta_{k-1} I)(T - \zeta_k I)x = 0$ and $(T - \zeta_k I)x \neq 0$. In this case, $(T - \zeta_k I)x$ is an eigenvector of T of eigenvalue ζ_{k-1}.
- xor $(T - \zeta_{k-2} I)(T - \zeta_{k-1} I)(T - \zeta_k I)x = 0$ and $(T - \zeta_{k-1} I)(T - \zeta_k I)x \neq 0$. In this case, $(T - \zeta_{k-1} I)(T - \zeta_k I)x$ is an eigenvector of T of eigenvalue ζ_{k-1}.
- ...
- xor $(T - \zeta_1 I) \dots (T - \zeta_k I)x = 0$ and $(T - \zeta_2 I) \dots (T - \zeta_k I)x = 0$. In this case, $(T - \zeta_2 I) \dots (T - \zeta_k I)x$ is an eigenvector of T of eigenvalue ζ_1.

For all of these cases, we find that T has an eigenvector v_1 and an eigenvalue λ_1.

As of now, it is possible for λ_1 to be complex. It is also possible for v_1 to be complex. (v_1 is of the form $(T - \zeta_i I)(T - \zeta_{k-1} I)(T - \zeta_k I)x$ where some ζ_i might be complex.) However we now prove that λ_1 must be real. And so must v_1 be. Since we have complex vectors and scalars, we must consider the complex inner product $\langle x, y \rangle_{\mathbb{C}}$ associated with $\langle x, y \rangle$.

We consider $\langle v_1, Tv_1 \rangle_{\mathbb{C}}$. One the one hand, we have

$$\langle v_1, Tv_1 \rangle_{\mathbb{C}} = \langle v_1, \lambda_1 v_1 \rangle_{\mathbb{C}} = \lambda_1 \langle v_1, v_1 \rangle_{\mathbb{C}}.$$

One the other hand, we have

$$\langle v_1, Tv_1 \rangle_{\mathbb{C}} = \langle Tv_1, v_1 \rangle_{\mathbb{C}} = \langle \lambda_1 v_1, v_1 \rangle_{\mathbb{C}} = \overline{\lambda_1} \langle v_1, v_1 \rangle_{\mathbb{C}}.$$

We get

$$\lambda_1 \langle v_1, v_1 \rangle_{\mathbb{C}} = \overline{\lambda_1} \langle v_1, v_1 \rangle_{\mathbb{C}}.$$

And, since $v_1 \neq 0$, $\langle v_1, v_1 \rangle_{\mathbb{C}} \neq 0$, so

$$\lambda_1 = \overline{\lambda_1}.$$

So

$$\lambda_1 \in \mathbb{R}.$$

(And then so is v_1.)

(d) We start by using part (c) to find an eigenvector v_1 and an associated (real) eigenvalue λ_1 of T.

We call $W_1 = \text{Span}(v_1)$. W_1 is invariant under T, so, by using part (b), we see that $W_1^\perp$ is invariant under T. Therefore we can consider T_1, the restriction of T to $W_1^\perp$. It is important to observe that, due to the invariance of $W_1^\perp$, T_1 is an operator. We have: $T_1 : W_1^\perp \mapsto W_1^\perp$. Since T_1 is an operator, it makes sense to speak about eigenvalues and eigenvectors for T_1. Also, any eigenvalues and eigenvectors of T_1 will also be eigenvalues and eigenvectors of T. It is also important to note that T_1

is self-adjoint. All this to say that, we can use part (c) on T_1 to find an eigenvector $v_2 \in W_1^\perp$ and an associated (real) eigenvalue λ_2 of T_1. This eigencouple (v_2, λ_2) of T_1 is also an eigencouple of T, and we have that v_1 and v_2 are mutually orthogonal eigenvectors of A.

We call $W_2 = \text{Span}(v_1, v_2)$. We note that v_1 and v_2 are two eigenvectors of T and there are mutually orthogonal. W_2 is invariant under T, (as any subspace spanned by eigenvectors of T is,) so, by using part (b), we see that $W_2^\perp$ is invariant under T. Therefore we can consider T_2, the restriction of T to $W_2^\perp$. It is important to observe that, due to the invariance of $W_2^\perp$, T_2 is an operator. We have: $T_2 : W_2^\perp \mapsto W_2^\perp$. Since T_2 is an operator, it makes sense to speak about eigenvalues and eigenvectors for T_2. Also, any eigenvalues and eigenvectors of T_2 will also be eigenvalues and eigenvectors of T. It is also important to note that T_2 is self-adjoint. All this to say that, we can use part (c) on T_2 to find an eigenvector $v_3 \in W_2^\perp$ and an associated (real) eigenvalue λ_3 of T_2. This eigencouple (v_3, λ_3) of T_2 is also an eigencouple of T, and we have v_1, v_2, and v_3 are mutually orthogonal eigenvectors of A.

We can continue in this manner until we find a basis of V made of n mutually orthogonal eigenvectors of A.

The last step is to normalize our n vectors so as to obtain an orthonormal basis of V made of eigenvectors of T.
