Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Each problem is worth 20 points
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of $n$-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space $V$. $T^*$ is the adjoint of the operator $T$ and $\lambda^*$ is the complex conjugate of the scalar $\lambda$. In an inner product space $V$, $U^\perp$ denotes the orthogonal complement of the subspace $U$.
- Ask the proctor if you have any questions.

Good luck!

1. __________  4. _________
2. __________  5. _________
3. __________  6. _________

Total _________

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Applied Linear Algebra Preliminary Exam Committee:
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Problem 1.

a. (6 points) Prove or reject:

There exists a matrix \( A \in \mathbb{R}^{4 \times 4} \) for which the column space and null space are identical.

Solution

We provide a matrix \( A \) that satisfies the claim. By the rank theorem, \( \text{rank}(A) + \dim \text{nul}(A) = 4 \). This implies that \( \text{rank}(A) = \dim \text{nul}(A) = 2 \), otherwise the column space and null space would be of different dimensions.

Particularly simple matrices \( A \) that satisfy the claim are

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\text{ or }
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

It is easy to see that then

\[
\text{col}(A) = \text{nul}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ or } \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},
\]

respectively. The design principle is to pick two of the columns of an identity matrix and put them in the column positions not used.

b. (9 points) Let \( A \neq 0 \) be an \( m \times n \) matrix with \( m \leq n \), let \( b \in \mathbb{R}^m \) such that \( Ax = b \) has no solution, and let \( d \neq 0 \in \mathbb{R}^m \) for which there exists a solution to \( Ax = d \).

What is the minimal and maximal dimension of the set of solutions for \( Ax = d \)? Provide the best bounds available based on the given information, prove that your bounds are correct, and prove that they can be tight for all well-defined \( m, n \).

Solution

As \( Ax = d \) has a solution, the dimension of its solution set is the same as \( \dim \text{nul}(A) \). Further note \( m \geq 2 \), otherwise the conditions \( A \neq 0 \) and \( Ax = b \) having no solution could not be satisfied at the same time. Recall that the null space is the orthogonal complement of the row space. By finding the possible \( \text{rank}(A) \), we also identify \( \dim \text{nul}(A) = n - \text{rank}(A) \).

As \( A \neq 0 \), one immediately obtains \( \text{rank}(A) \geq 1 \) and thus \( \dim \text{nul}(A) \leq n - 1 \). For an upper bound, first recall the trivial bound \( \text{rank}(A) \leq \min\{m, n\} \). As \( m \leq n \), this simplifies to \( \text{rank}(A) \leq m \). However, this is not the best bound possible yet: As there exists a \( b \) for which the system \( Ax = b \) is inconsistent, we know that
there is a row of all zeros in the unique reduced echelon form matrix $B$ that is row-equivalent to $A$. This implies that $\text{rank}(A) \leq m - 1$. Thus $\text{dim nul}(A) \geq n - m + 1$. Together, one obtains the bounds

$$n - m + 1 \leq \text{dim nul}(A) \leq n - 1,$$

which is well-defined as $m \geq 2$.

To prove that these are the best bounds available for the given information, one should provide $m \times n$ matrices of rank 1 and rank $m - 1$ for all $2 \leq m \leq n$, for example

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix},$$

as well as a right-hand side $b$ that differs in the first two entries, and a right-hand side $d$ where these entries are the same.

c. (5 points) Suppose that $S$ is a fixed, invertible $n \times n$ matrix. Let $W$ be the set of all matrices $A$ for which $S^{-1}AS$ is diagonal.

Prove or reject: $W$ is a vector space.

**Solution**

Let $W = \{ A \in \mathbb{R}^{n\times n} : S^{-1}AS \text{ is diagonal} \}$. Recall $\mathbb{R}^{n\times n}$ is a vector space itself, so it suffices to show that $W$ is a subspace of it. To do so, we have to check whether $0 \in W$ and whether $W$ is closed under addition and scaling.

- $0 \in W$ is a diagonal matrix, as $S^{-1}0S = 0$.
- Let $A, B \in W$. Then $S^{-1}(A + B)S = S^{-1}AS + S^{-1}BS$ and both parts of this sum are diagonal. Then so is their sum, which shows $A + B \in W$.
- Let $A \in W$ and $c \in \mathbb{R}$. Then $S^{-1}(cA)S = cS^{-1}AS$, which is diagonal because $S^{-1}AS$ is diagonal. This shows $cA \in W$. 

Problem 2.

a. (4 points) Let \( T : \mathbb{P}_3 \rightarrow \mathbb{P}_3 \) be an operator that maps \( p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \) onto \( q(t) = a_3 t^1 + a_2 t^2 + a_1 t^3 \).

Prove or reject: \( T \) is a linear transformation. If so, provide a matrix representation.

**Solution**

Using the standard basis of monomials of \( \mathbb{P}_3 \), the provided information can be written using coordinate vectors as

\[
T \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix}.
\]

It is easy to verify that

\[
\begin{pmatrix} 0 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix},
\]

which is a matrix representation of \( T \). The ability to provide such a representation immediately implies that \( T \) is a linear transformation.

b. (7 points) The first four Hermite polynomials are

\( 1, \quad 1 - t, \quad -2 + 4t^2, \quad -12t + 18t^3 \).

They form a basis \( \beta \) of \( \mathbb{P}_3 \), the space of polynomials of degree at most 3.

Compute the change-of-coordinates matrix \( P_{\beta \rightarrow \gamma} \) from \( \beta \) to a new basis \( \gamma \) of \( \mathbb{P}_3 \) given by

\( t^3 + t^2 + 2t, \quad t^2 + 2t, \quad 1 + t, \quad t \).

(Hint: \( P_{\beta \rightarrow \gamma} \), when multiplied with a coordinate vector with respect to \( \beta \) gives a coordinate vector with respect to \( \gamma \).)

**Solution**

Let \( E \) denote the standard basis of monomials of \( \mathbb{P}_3 \). Then

\[
P_{\beta \rightarrow E} = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 18 \end{pmatrix} \quad \text{and} \quad P_{\gamma \rightarrow E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.
Now, note $P_{β→γ} = P_{E→γ} \cdot P_{β→E}$ and $P_{E→γ} = P_{γ→E}^{-1}$. In a short computation, we invert $P_{γ→E}$ to find

$$P_{E→γ} = P_{γ→E}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 \end{pmatrix},$$

and finally

$$P_{β→γ} = P_{E→γ} \cdot P_{β→E} = \begin{pmatrix} 0 & 0 & 0 & 18 \\ 0 & 0 & 4 & -18 \\ 1 & 1 & -2 & 0 \\ -1 & -2 & -6 & -12 \end{pmatrix}.$$

c. (9 points) Let $a, b \neq 0 \in \mathbb{R}$ be fixed. Find a basis for the subspace in $\mathbb{R}^4$ created from intersecting

$$S = \text{span} \left\{ \begin{pmatrix} a \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ b \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad T = \text{span} \left\{ \begin{pmatrix} b \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ b \end{pmatrix} \right\}.$$  

**Solution**

First, note that scaling any spanning vectors does not change the span. So $S$ and $T$ can be represented using $c = \frac{b}{a}$ as

$$S = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad T = \text{span} \left\{ \begin{pmatrix} c \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$ 

An element $x \in \mathbb{R}^4$ belongs to the intersection $S \cap T$ if and only if

$$x = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_3 \\ \lambda_2 \\ \lambda_1 \end{pmatrix} = \mu_1 \begin{pmatrix} c \\ 0 \\ 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ c \mu_2 \\ \mu_1 \end{pmatrix},$$

for scalars $λ_1,2,3$ and $μ_1,2$. It follows that $λ_1 = μ_1$, $λ_2 = cμ_2$, $λ_3 = μ_2$, and $λ_1 + λ_2 + λ_3 = cμ_1$. 

We now assume that $x$ is given through $μ_{1,2}$ (and the fixed $c$), and identify when the above linear system has a solution. The augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & \mu_1 \\ 0 & 1 & 0 & cμ_2 \\ 0 & 0 & 1 & \mu_2 \\ 1 & 1 & 1 & cμ_1 \end{pmatrix}$$
which reduces to

\[
\begin{pmatrix}
  1 & 0 & 0 & \mu_1 \\
  0 & 1 & 0 & c \mu_2 \\
  0 & 0 & 1 & \mu_2 \\
  0 & 0 & 0 & (c - 1) \mu_1 - (c + 1) \mu_2
\end{pmatrix}.
\]

This system is solvable if and only if

\[(c - 1) \mu_1 - (c + 1) \mu_2 = 0 \iff (c - 1) \mu_1 = (c + 1) \mu_2.\]

If \(c \neq 1\), this is equivalent to \(\mu_1 = \frac{c + 1}{c - 1} \mu_2\). This gives

\[
S \cap T = \text{span}\left\{ \begin{pmatrix}
\frac{c+1}{c-1} \\
1 \\
c \\
\frac{c+1}{c-1}
\end{pmatrix} \right\}.
\]

Otherwise, that is if \(c = 1\), then \(\mu_2 = 0\) and one obtains

\[
S \cap T = \text{span}\left\{ \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix} \right\}.
\]
Problem 3.

Let $V$ be a finite-dimensional vector space.

a. (7 points) Suppose $T \in \mathcal{L}(V)$ is such that every vector in $V$ is an eigenvector of $T$. Prove or disprove that $T$ is a scalar multiple of the identity operator.

Solution

If $\dim V \leq 1$, every linear operator on $V$ is a scalar multiple of the identity operator, so there is nothing to prove. Otherwise, suppose $u$ and $v$ are two linearly independent vectors in $V$. Since all vectors in $V$ are eigenvectors, there exist scalars $\alpha, \beta$ and $\gamma$ such that

\[ Tu = \alpha u, \quad Tv = \beta v, \quad T(u + v) = \gamma (u + v). \]

But, $T(u + v) = Tu + Tv = \alpha u + \beta v$, so $(\gamma - \alpha)u + (\gamma - \beta)v = 0$. This implies $\alpha = \beta = \gamma$ since $u$ and $v$ are linearly independent. Thus, $T$ has only one eigenvalue, $\alpha$. Thus, $Tv = \alpha v$ for all $v \in V$, so $T$ is a scalar multiple of the identity operator.

b. (13 points) Suppose $T \in \mathcal{L}(V)$ is such that every subspace of $V$ with dimension $\dim V - 1$ is invariant under $T$. Prove that $T$ is a scalar multiple of the identity operator.

Solution

Suppose $v \in V$ is not an eigenvector of $T$ and let $u = Tv$. Since $v$ is not an eigenvector, $u$ and $v$ are linearly independent. Thus, $\{u, v\}$ can be extended to a basis $\{u, v, w_3, \ldots, w_n\}$ of $V$. Let $W = \text{span} \{v, w_3, \ldots, w_n\}$. Observe that $\dim W = \dim V - 1$, so $W$ is invariant under $T$. Since $v \in W$, it follows that $Tv = u \in W$, which is a contradiction. Thus, every vector in $V$ is an eigenvector, so by part a), $T$ is a scalar multiple of the identity operator.
Problem 4.

Let $\| \cdot \|$ denote an arbitrary vector norm on $\mathbb{R}^p$. The matrix norm induced by $\| \cdot \|$ is defined by

$$\| P \| = \max_{x \neq 0} \frac{\| Px \|}{\| x \|}$$

for each $p \times p$ real matrix $P$.

a. (7 points) Prove that $\| \cdot \|$ is a norm on the vector space of real $p \times p$ matrices.

Solution

We need to verify that the induced norm satisfies the three properties of norms:

1) $\| P \| > 0$ for $P \neq 0$; 2) for any scalar $\alpha$ and matrix $P$, $\| \alpha P \| = |\alpha| \| P \|$ and 3) for any two matrix $P$ and $Q$, $\| P \| + \| Q \| \leq \| P + Q \|$.

1) Since $\| \cdot \|$ is a vector norm, $\| Px \| \geq 0$ for all $P$ and $x$. Thus, the right hand side in the definition above is always nonnegative, so $\| P \| \geq 0$. Moreover, if $P \neq 0$, it has rank $\geq 1$; thus, we can find $\bar{x} \in \mathbb{R}^p$ such that $P \bar{x} \neq 0$. But then $\| P \| \geq \| P \bar{x} \| \| \bar{x} \| > 0$. Thus, $\| P \| > 0$ for all $P \neq 0$.

2) For any scalar $\alpha$ we have

$$\| \alpha P \| = \max_{x \neq 0} \frac{\| \alpha Px \|}{\| x \|} = \max_{x \neq 0} \frac{|\alpha| \| Px \|}{\| x \|} = |\alpha| \max_{x \neq 0} \frac{\| Px \|}{\| x \|} = |\alpha| \| P \|.$$

3) For two matrices $P$ and $Q$, we have

$$\| P + Q \| = \max_{x \neq 0} \frac{\| (P + Q)x \|}{\| x \|} \leq \max_{x \neq 0} \frac{\| Px \| + \| Qx \|}{\| x \|} \leq \max_{x \neq 0} \frac{\| Px \|}{\| x \|} + \max_{y \neq 0} \frac{\| Qy \|}{\| y \|} = \| P \| + \| Q \|.$$

b. (13 points) Let $P$ be a $p \times p$ real matrix. Suppose that $\| P \| < 1$. Prove that $I + P$ is nonsingular and that

$$\frac{1}{1 + \| P \|} \leq \| (I + P)^{-1} \| \leq \frac{1}{1 - \| P \|}.$$ 

Solution

Suppose $x$ is a solution to the equation $(I + P)x = 0$. Then $x = -Px$, so

$$\| x \| = \| -Px \| \leq \| P \| \| x \|.$$
Since $\|P\| < 1$, this implies that $x = 0$. (Otherwise, we get the contradiction $\|x\| < \|x\|$). Thus, the only solution to $(I + P)x = 0$ is the trivial solution $x = 0$, so $I + P$ is nonsingular.

Let $B = (I + P)^{-1}$. Then $I = B(I + P)$. Thus,

$$1 = \|I\| = \|B(I + P)\| \leq \|B\|\|I + P\| \leq \|B\|(1 + \|P\|).$$

Thus,

$$\frac{1}{1 + \|P\|} \leq \|B\| = \|(I + P)^{-1}\|.$$

To get the second inequality, observe that $I = B + BP$, so $B = I - BP$. Thus,

$$\|B\| = \|I - BP\| \leq 1 + \|BP\| \leq 1 + \|B\|\|P\|.$$

Hence, $\|B\|(1 - \|P\|) \leq 1$ and $\|B\| \leq \frac{1}{1 - \|P\|}$. 
Problem 5.

Let $V$ be an $n$-dimensional inner product space over $\mathbb{F}$.

a. (5 points) Suppose $T \in \mathcal{L}(V)$ and $U$ is a subspace of $V$. Prove or reject: $U^\perp$ is invariant under $T^*$ if $U$ is invariant under $T$.

**Solution**

Suppose $U$ is invariant under $T$. To show $U^\perp$ is invariant under $T^*$, let $v \in U^\perp$, and then

$$\langle u, T^* v \rangle = \langle Tu, v \rangle = 0$$

for every $u \in U$, since $Tu \in U$. So $T^* v \in U^\perp$. So $U^\perp$ is invariant under $T^*$.

b. (5 points) Let $T_1$ and $T_2$ be two self-adjoint operators on $V$. Prove or reject: $T_1 T_2 + T_2 T_1$ is also self-adjoint.

**Solution**

For any $u \in V, v \in V$,

$$\langle (T_1 T_2 + T_2 T_1) u, v \rangle = \langle T_1 T_2 u + T_2 T_1 u, v \rangle = \langle T_1 T_2 u, v \rangle + \langle T_2 T_1 u, v \rangle$$

$$= \langle T_2 u, T_1^* v \rangle + \langle T_1 u, T_2^* v \rangle = \langle T_2 u, T_1^* v \rangle + \langle T_1 u, T_2^* v \rangle$$

$$= \langle u, T_2 T_1^* v \rangle + \langle u, T_1 T_2^* v \rangle = \langle u, T_2 T_1^* v \rangle + \langle u, T_1 T_2^* v \rangle$$

$$= \langle u, T_2 T_1^* v + T_1 T_2^* v \rangle = \langle u, (T_2 T_1 + T_1 T_2)^* v \rangle$$

So $T_2 T_1 + T_1 T_2$ is self-adjoint.

c. (10 points) Let $T$ be a self-adjoint operator on $V$. Show that $T$ is a nonnegative self-adjoint operator on $V$ if and only if the eigenvalues of $T$ are all nonnegative real numbers.

**Solution**

Since $T$ is self-adjoint, all of its eigenvalues are real.

"⇒": Suppose $T$ is nonnegative and self-adjoint. Let $\lambda$ be an eigenvalue of $T$, with corresponding eigenvector $v \neq 0$. Then

$$Tv = \lambda v, \quad v \neq 0$$

and

$$\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle \geq 0$$

Since $\langle v, v \rangle > 0$, $\lambda \geq 0$. 
Since $T$ is self-adjoint, by the Spectral Theorem, there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $V$ whose basis vectors are eigenvectors of $V$:

$$
Te_1 = \lambda_1 e_1 \\
Te_2 = \lambda_2 e_2 \\
\vdots \\
Te_n = \lambda_n e_n
$$

where $\lambda_i, i = 1, \ldots, n$ are all the eigenvalues of $T$.

For any vector $v \in V$,

$$
v = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n
$$

Then

$$
\langle Tv, v \rangle = \langle T(c_1 e_1 + c_2 e_2 + \cdots + c_n e_n), c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \rangle \\
= \langle c_1 Te_1 + c_2 Te_2 + \cdots + c_n Te_n, c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \rangle \\
= \langle c_1 \lambda_1 e_1 + c_2 \lambda_2 e_2 + \cdots + c_n \lambda_n e_n, c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \rangle \\
= c_1^2 \lambda_1 + c_2^2 \lambda_2 + \cdots + c_n^2 \lambda_n
$$

Since $\lambda_i \geq 0, i = 1, \ldots, n$, the above quantity is nonnegative. So $T$ is nonnegative.
Problem 6.

a. (6 points) Let \( A \in \mathbb{F}^{n,n} \) be a square matrix that satisfies \( A^2 = A \). Show that \( A \) is similar to the diagonal matrix.

\[
    C = \begin{bmatrix}
        1 &  &  & \\
        & \ddots &  & \\
        &  & 1 &  \\
        &  &  & 0 \\
    \end{bmatrix} = \begin{bmatrix}
        \mathcal{I}_r &  \\
        0 &  \\
    \end{bmatrix}
\]

That is, \( \mathcal{I}_r \) is an identity square block of order \( r \), \( 0 \leq r \leq n \).

**Solution**

Let \( f(\lambda) = \lambda^2 - \lambda \). Then

\[
    f(A) = A^2 - A = 0
\]

So the minimal polynomial of \( A \) divides \( f \). So the eigenvalues can only be 0 or 1, and each Jordan block is of size \( 1 \times 1 \). Rearranging the diagonal elements in the Jordan canonical form, we have \( A \) is similar to \( C \).

b. (6 points) Let \( A \in \mathbb{F}^{n,n}, B \in \mathbb{F}^{n,n} \) be square matrices such that \( A^2 = A, B^2 = B, \) and \( AB = BA \). Suppose \( P_0 \) is an invertible matrix such that

\[
    P_0^{-1}AP_0 = \begin{bmatrix}
        \mathcal{I}_r &  \\
        0 &  \\
    \end{bmatrix}
\]

Let \( B_0 = P_0^{-1}BP_0 \). Show that \( B_0 \) is in the form of

\[
    B_0 = \begin{bmatrix}
        B_1 &  \\
        B_2 &  \\
    \end{bmatrix}
\]

where \( B_1 \) is of order \( r \), and \( B_1^2 = B_1 \) and \( B_2^2 = B_2 \).

**Solution**

Note

\[
    AB = BA \iff P_0 \begin{bmatrix}
        \mathcal{I}_r &  \\
        0 &  \\
    \end{bmatrix} P_0^{-1} P_0 B_0 P_0^{-1} = P_0 B_0 P_0^{-1} \begin{bmatrix}
        \mathcal{I}_r &  \\
        0 &  \\
    \end{bmatrix} P_0^{-1}
\]

which is equivalent to

\[
    B_0 \begin{bmatrix}
        \mathcal{I}_r &  \\
        0 &  \\
    \end{bmatrix} = \begin{bmatrix}
        \mathcal{I}_r &  \\
        0 &  \\
    \end{bmatrix} B_0
\]
So we can conclude that
\[ B_0 = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

In addition,
\[ B_0^2 = (P_0^{-1}BP_0)(P_0^{-1}BP_0) = P_0^{-1}B^2P_0 = P_0^{-1}BP_0 = B_0 \]

So
\[ \begin{bmatrix} B_1^2 \\ B_2^2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \]

So \( B_1^2 = B_1 \) and \( B_2^2 = B_2 \).

c. (8 points) Let \( A \in F^{n,n} \), \( B \in F^{n,n} \) be square matrices such that \( A^2 = A \), \( B^2 = B \), and \( AB = BA \). Show that there exists an invertible matrix \( P \) such that \( P^{-1}AP \) and \( P^{-1}BP \) are both diagonal, and the diagonal entries are 0 and 1 for both.

(Hint: Let \( P_0 \) be the invertible matrix for \( A \) in part (b). Let \( Q_1 \) and \( Q_2 \) be invertible matrices that serve the same role for \( B_1 \) and \( B_2 \), respectively. Use \( P_0, Q_1 \) and \( Q_2 \) to construct the matrix \( P \).)

**Solution**

Since the \( B_1 \) and \( B_2 \) from (b) satisfy \( B_1^2 = B_1 \) and \( B_2^2 = B_2 \), there exist invertible matrices \( Q_1 \) and \( Q_2 \) such that

\[ Q_1^{-1}B_1Q_1 = \begin{bmatrix} I_s \\ 0 \end{bmatrix}, s \leq r \]

and

\[ Q_2^{-1}B_2Q_2 = \begin{bmatrix} I_t \\ 0 \end{bmatrix}, t \leq n - r \]

Let

\[ Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad P = P_0Q \]

Then

\[
P^{-1}BP = P^{-1}(P_0BP_0^{-1})P = Q^{-1}P_0^{-1}(P_0BP_0^{-1})P_0Q = Q^{-1}B_0Q \]

\[
= \begin{bmatrix} Q_1^{-1} & Q_2^{-1} \\ Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_1^{-1}B_1Q_1 \\ Q_2^{-1}B_2Q_2 \end{bmatrix} \]

\[
= \begin{bmatrix} I_s \\ 0 \\ I_t \\ 0 \end{bmatrix}
\]
On the other hand

\[ P^{-1}AP = Q^{-1}P_0^{-1}AP_0Q \]

\[ = \begin{bmatrix} Q_1^{-1} & Q_2^{-2} \end{bmatrix} \begin{bmatrix} I_r & 0 \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} I_r & 0 \end{bmatrix} \]

So we have found the matrix \( P \).