University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
July 12, 2019

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to complete all six problems.
- Each problem is worth 20 points
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Notation: Throughout the exam, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers, respectively. $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. $\mathbb{F}^n$ and $\mathbb{F}^{n,n}$ are the vector spaces of n-tuples and $n \times n$ matrices, respectively, over the field $\mathbb{F}$. $\mathcal{L}(V)$ denotes the set of linear operators on the vector space V. T^* is the adjoint of the operator T and λ^* is the complex conjugate of the scalar λ. In an inner product space V, $U^\perp$ denotes the orthogonal complement of the subspace U.
- Ask the proctor if you have any questions.

Good luck!

1.	________	4.	________
2.	________	5.	________
3.	________	6.	________
		Total	________

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Applied Linear Algebra Preliminary Exam Committee:
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Problem 1.

a. (6 points) Prove or reject:

There exists a matrix $A \in \mathbb{R}^{4\times 4}$ for which the column space and null space are identical.

Solution

We provide a matrix A that satisfies the claim. By the rank theorem, $\operatorname{rank}(A) + \dim \operatorname{nul}(A) = 4$. This implies that $\operatorname{rank}(A) = \dim \operatorname{nul}(A) = 2$, otherwise the column space and null space would be of different dimensions.

Particularly simple matrices A that satisfy the claim are

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that then

$$\operatorname{col}(A) = \operatorname{nul}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{or} \quad = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

respectively. The design principle is to pick two of the columns of an identity matrix and put them in the column positions not used.

b. (9 points) Let $A \neq 0$ be an $m \times n$ matrix with $m \leq n$, let $b \in \mathbb{R}^m$ such that $Ax = b$ has no solution, and let $d \neq 0 \in \mathbb{R}^m$ for which there exists a solution to $Ax = d$.

What is the minimal and maximal dimension of the set of solutions for $Ax = d$? Provide the best bounds available based on the given information, prove that your bounds are correct, and prove that they can be tight for all well-defined m, n.

Solution

As $Ax = d$ has a solution, the dimension of its solution set is the same as $\dim \operatorname{nul}(A)$. Further note $m \geq 2$, otherwise the conditions $A \neq 0$ and $Ax = b$ having no solution could not be satisfied at the same time. Recall that the null space is the orthogonal complement of the row space. By finding the possible $\operatorname{rank}(A)$, we also identify $\dim \operatorname{nul}(A) = n - \operatorname{rank}(A)$.

As $A \neq 0$, one immediately obtains $\operatorname{rank}(A) \geq 1$ and thus $\dim \operatorname{nul}(A) \leq n - 1$. For an upper bound, first recall the trivial bound $\operatorname{rank}(A) \leq \min\{m, n\}$. As $m \leq n$, this simplifies to $\operatorname{rank}(A) \leq m$. However, this is not the best bound possible yet: As there exists a b for which the system $Ax = b$ is inconsistent, we know that there is a row of all zeros in the unique reduced echelon form matrix B that is row-equivalent to A. This implies that $\text{rank}(A) \leq m-1$. Thus $\dim \text{nul}(A) \geq n-m+1$. Together, one obtains the bounds

$$n - m + 1 \leq \dim \text{nul}(A) \leq n - 1,$$

which is well-defined as $m \geq 2$.

To prove that these are the best bounds available for the given information, one should provide $m \times n$ matrices of rank 1 and rank $m-1$ for all $2 \leq m \leq n$, for example

$$\begin{pmatrix} 1 & 0 & ... & 0 \\ 1 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & ... & 0 & 0 & ... & 0 \\ 1 & 0 & ... & 0 & 0 & ... & 0 \\ 0 & 1 & ... & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & 0 & ... & 0 \end{pmatrix},$$

as well as a right-hand side b that differs in the first two entries, and a right-hand side d where these entries are the same.

c. (5 points) Suppose that S is a fixed, invertible $n \times n$ matrix. Let W be the set of all matrices A for which $S^{-1}AS$ is diagonal.

Prove or reject: W is a vector space.

Solution

Let $W = \{A \in \mathbb{R}^{n \times n} : S^{-1}AS \text{ is diagonal }\}$. Recall $\mathbb{R}^{n \times n}$ is a vector space itself, so it suffices to show that W is a subspace of it. To do so, we have to check whether $0 \in W$ and whether W is closed under addition and scaling.

- $0 \in W$ is a diagonal matrix, as $S^{-1}0S = 0$.
- Let $A, B \in W$. Then $S^{-1}(A+B)S = S^{-1}AS + S^{-1}BS$ and both parts of this sum are diagonal. Then so is their sum, which shows $A + B \in W$.
- Let $A \in W$ and $c \in \mathbb{R}$. Then $S^{-1}(cA)S = cS^{-1}AS$, which is diagonal because $S^{-1}AS$ is diagonal. This shows $cA \in W$.

Problem 2.

a. (4 points) Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be an operator that maps $p(t) = a_0 + a_1 t^1 + a_2 t^2 + a_3 t^3$ onto $q(t) = a_3 t^1 + a_2 t^2 + a_1 t^3$.

Prove or reject: T is a linear transformation. If so, provide a matrix representation.

Solution

Using the standard basis of monomials of $\mathbb{P}_3$, the provided information can be written using coordinate vectors as

$$T\left(\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix}.$$

It is easy to verify that

$$\begin{pmatrix} 0 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

which is a matrix representation of T. The ability to provide such a representation immediately implies that T is a linear transformation.

b. (7 points) The first four Hermite polynomials are

$$1, \quad 1 - t, \quad -2 + 4t^2, \quad -12t + 18t^3.$$

They form a basis β of $\mathbb{P}_3$, the space of polynomials of degree at most 3.

Compute the change-of-coordinates matrix $P_{\beta \to \gamma}$ from β to a new basis γ of $\mathbb{P}_3$ given by

$$t^3 + t^2 + 2t, \quad t^2 + 2t, \quad 1 + t, \quad t.$$

(Hint: $P_{\beta \to \gamma}$, when multiplied with a coordinate vector with respect to β gives a coordinate vector with respect to γ.)

Solution

Let E denote the standard basis of monomials of $\mathbb{P}_3$. Then

$$P_{\beta \to E} = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & -1 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 18 \end{pmatrix} \quad \text{and} \quad P_{\gamma \to E} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Now, note $P_{\beta\to\gamma} = P_{E\to\gamma} \cdot P_{\beta\to E}$ and $P_{E\to\gamma} = P^{-1}_{\gamma\to E}$. In a short computation, we invert $P_{\gamma\to E}$ to find

$$P_{E\to\gamma} = P^{-1}_{\gamma\to E} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 \end{pmatrix},$$

and finally

$$P_{\beta\to\gamma} = P_{E\to\gamma} \cdot P_{\beta\to E} = \begin{pmatrix} 0 & 0 & 0 & 18 \\ 0 & 0 & 4 & -18 \\ 1 & 1 & -2 & 0 \\ -1 & -2 & -6 & -12 \end{pmatrix}.$$

c. (9 points) Let $a, b \neq 0 \in \mathbb{R}$ be fixed. Find a basis for the subspace in $\mathbb{R}^4$ created from intersecting

$$S = \text{span}\left\{ \begin{pmatrix} a \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} a \\ 0 \\ a \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ b \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad T = \text{span}\left\{ \begin{pmatrix} b \\ 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 \\ a \\ b \\ 0 \end{pmatrix} \right\}.$$

Solution

First, note that scaling any spanning vectors does not change the span. So S and T can be represented using $c = \frac{b}{a}$ as

$$S = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad T = \text{span}\left\{ \begin{pmatrix} c \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ c \\ 0 \end{pmatrix} \right\}.$$

An element $x \in \mathbb{R}^4$ belongs to the intersection $S \cap T$ if and only if

$$x = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 + \lambda_3 \\ \lambda_3 \\ \lambda_2 \\ \lambda_1 \end{pmatrix} = \mu_1 \begin{pmatrix} c \\ 0 \\ 0 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ 1 \\ c \\ 0 \end{pmatrix} = \begin{pmatrix} c\mu_1 \\ \mu_2 \\ c\mu_2 \\ \mu_1 \end{pmatrix},$$

for scalars $\lambda_{1,2,3}$ and $\mu_{1,2}$. It follows that $\lambda_1 = \mu_1$, $\lambda_2 = c\mu_2$, $\lambda_3 = \mu_2$, and $\lambda_1 + \lambda_2 + \lambda_3 = c\mu_1$.

We now assume that x is given through $\mu_{1,2}$ (and the fixed c), and identify when the above linear system has a solution. The augmented matrix is

$$\begin{pmatrix} 1 & 0 & 0 & \mu_1 \\ 0 & 1 & 0 & c\mu_2 \\ 0 & 0 & 1 & \mu_2 \\ 1 & 1 & 1 & c\mu_1 \end{pmatrix}$$

which reduces to

$$\begin{pmatrix} 1 & 0 & 0 & \mu_1 \\ 0 & 1 & 0 & c\mu_2 \\ 0 & 0 & 1 & \mu_2 \\ 0 & 0 & 0 & (c-1)\mu_1 - (c+1)\mu_2 \end{pmatrix}.$$

This system is solvable if and only if

$$(c-1)\mu_1 - (c+1)\mu_2 = 0 \Leftrightarrow (c-1)\mu_1 = (c+1)\mu_2.$$

If $c \neq 1$, this is equivalent to $\mu_1 = \frac{c+1}{c-1}\mu_2$. This gives

$$S \cap T = \text{span}\left\{ \begin{pmatrix} \frac{c+1}{c-1}c \\ 1 \\ c \\ \frac{c+1}{c-1} \end{pmatrix} \right\}.$$

Otherwise, that is if $c = 1$, then $\mu_2 = 0$ and one obtains

$$S \cap T = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Problem 3.

Let V be a finite-dimensional vector space.

a. (7 points) Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove or disprove that T is a scalar multiple of the identity operator.

   Solution

   If $\dim V \leq 1$, every linear operator on V is a scalar multiple of the identity operator, so there is nothing to prove. Otherwise, suppose u and v are two linearly independent vectors in V. Since all vectors in V are eigenvectors, there exist scalars α, β and γ such that

   $$Tu = \alpha u, \ Tv = \beta v, \ \text{and} \ T(u+v) = \gamma(u+v).$$

   But, $T(u+v) = Tu + Tv = \alpha u + \beta v$, so $(\gamma - \alpha)u + (\gamma - \beta)v = 0$. This implies $\alpha = \beta = \gamma$ since u and v are linearly independent. Thus, T has only one eigenvalue, α. Thus, $Tv = \alpha v$ for all $v \in V$, so T is a scalar multiple of the identity operator.

b. (13 points) Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

   Solution

   Suppose $v \in V$ is not an eigenvector of T and let $u = Tv$. Since v is not an eigenvector, u and v are linearly independent. Thus, $\{u, v\}$ can be extended to a basis $\{u, v, w_3, \ldots, w_n\}$ of V. Let $W = \operatorname{span}\{v, w_3, \ldots, w_n\}$. Observe that $\dim W = \dim V - 1$, so W is invariant under T. Since $v \in W$, it follows that $Tv = u \in W$, which is a contradiction. Thus, every vector in V is an eigenvector, so by part a), T is a scalar multiple of the identity operator.

Problem 4.

Let $\|\cdot\|$ denote an arbitrary vector norm on $\mathbb{R}^p$. The matrix norm induced by $\|\cdot\|$ is defined by

$$\|P\| = \max_{x \neq 0} \frac{\|Px\|}{\|x\|}$$

for each $p \times p$ real matrix P.

a. (7 points) Prove that $\|\cdot\|$ is a norm on the vector space of real $p \times p$ matrices.

Solution

We need to verify that the induced norm satisfies the three properties of norms: 1) $\|P\| > 0$ for $P \neq 0$; 2) for any scalar α and matrix P, $\|\alpha P\| = |\alpha|\|P\|$ and 3) for any two matrix P and Q, $\|P\| + \|Q\| \leq \|P\| + \|Q\|$.

1) Since $\|\cdot\|$ is a vector norm, $\|Px\| \geq 0$ for all P and x. Thus, the right hand side in the definition above is always nonnegative, so $\|P\| \geq 0$. Moreover, if $P \neq 0$, it has rank ≥ 1; thus, we can find $\bar{x} \in \mathbb{R}^p$ such that $P\bar{x} \neq 0$. But then $\|P\| \geq \frac{\|P\bar{x}\|}{\|x\|} > 0$. Thus, $\|P\| > 0$ for all $P \neq 0$.

2) For any scalar α we have

$$\|\alpha P\| = \max_{x \neq 0} \frac{\|\alpha Px\|}{\|x\|} = \max_{x \neq 0} \frac{|\alpha|\|Px\|}{\|x\|} = |\alpha| \max_{x \neq 0} \frac{\|Px\|}{\|x\|} = |\alpha|\|P\|.$$

3) For two matrices P and Q, we have

$$\begin{aligned}\|P+Q\| = \max_{x \neq 0} \frac{\|(P+Q)x\|}{\|x\|} &\leq \max_{x \neq 0} \frac{\|Px\| + \|Qx\|}{\|x\|} \\ &\leq \max_{x \neq 0} \frac{\|Px\|}{\|x\|} + \max_{y \neq 0} \frac{\|Qy\|}{\|y\|} = \|P\| + \|Q\|\end{aligned}$$

b. (13 points) Let P be a $p \times p$ real matrix. Suppose that $\|P\| < 1$. Prove that $I + P$ is nonsingular and that

$$\frac{1}{1 + \|P\|} \leq \|(I+P)^{-1}\| \leq \frac{1}{1 - \|P\|}.$$

Solution

Suppose x is a solution to the equation $(I+P)x = 0$. Then $x = -Px$, so

$$\|x\| = \|-Px\| \leq \|P\|\|x\|.$$

Since $\|P\| < 1$, this implies that $x = 0$. (Otherwise, we get the contradiction $\|x\| < \|x\|$). Thus, the only solution to $(I + P)x = 0$ is the trivial solution $x = 0$, so $I + P$ is nonsingular.

Let $B = (I + P)^{-1}$. Then $I = B(I + P)$. Thus,

$$1 = \|I\| = \|B(I + P)\| \leq \|B\|\|I + P\| \leq \|B\|(1 + \|P\|).$$

Thus,

$$\frac{1}{1 + \|P\|} \leq \|B\| = \|(I + P)^{-1}\|.$$

To get the second inequality, observe that $I = B + BP$, so $B = I - BP$. Thus,

$$\|B\| = \|I - BP\| \leq 1 + \|BP\| \leq 1 + \|B\|\|P\|.$$

Hence, $\|B\|(1 - \|P\|) \leq 1$ and $\|B\| \leq \frac{1}{1-\|P\|}$.

Problem 5.

Let V be an n-dimensional inner product space over $\mathbb{F}$.

a. (5 points) Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove or reject: $U^{\perp}$ is invariant under T^* if U is invariant under T.

Solution

Suppose U is invariant under T. To show $U^{\perp}$ is invariant under T^*, let $\boldsymbol{v} \in U^{\perp}$, and then

$$\langle \boldsymbol{u}, T^*\boldsymbol{v} \rangle = \langle T\boldsymbol{u}, \boldsymbol{v} \rangle = 0$$

for every $\boldsymbol{u} \in U$, since $T\boldsymbol{u} \in U$. So $T^*\boldsymbol{v} \in U^{\perp}$. So $U^{\perp}$ is invariant under T^*.

b. (5 points) Let T_1 and T_2 be two self-adjoint operators on V. Prove or reject: $T_1T_2 + T_2T_1$ is also self-adjoint.

Solution

For any $\boldsymbol{u} \in V, \boldsymbol{v} \in V$,

$$\begin{aligned}
\langle (T_1T_2 + T_2T_1)\boldsymbol{u}, \boldsymbol{v} \rangle &= \langle T_1T_2\boldsymbol{u} + T_2T_1\boldsymbol{u}, \boldsymbol{v} \rangle = \langle T_1T_2\boldsymbol{u}, \boldsymbol{v} \rangle + \langle T_2T_1\boldsymbol{u}, \boldsymbol{v} \rangle \\
&= \langle T_2\boldsymbol{u}, T_1^*\boldsymbol{v} \rangle + \langle T_1\boldsymbol{u}, T_2^*\boldsymbol{v} \rangle = \langle T_2\boldsymbol{u}, T_1\boldsymbol{v} \rangle + \langle T_1\boldsymbol{u}, T_2\boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u}, T_2^*T_1\boldsymbol{v} \rangle + \langle \boldsymbol{u}, T_1^*T_2\boldsymbol{v} \rangle = \langle \boldsymbol{u}, T_2T_1\boldsymbol{v} \rangle + \langle \boldsymbol{u}, T_1T_2\boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u}, T_2T_1\boldsymbol{v} + T_1T_2\boldsymbol{v} \rangle = \langle \boldsymbol{u}, (T_2T_1 + T_1T_2)\boldsymbol{v} \rangle
\end{aligned}$$

So $T_2T_1 + T_1T_2$ is self-adjoint.

c. (10 points) Let T be a self-adjoint operator on V. Show that T is a nonnegative self-adjoint operator on V if and only if the eigenvalues of T are all nonnegative real numbers.

Solution

Since T is self-adjoint, all of its eigenvalues are real.

"$\Rightarrow$": Suppose T is nonnegative and self-adjoint. Let λ be an eigenvalue of T, with corresponding eigenvector $\boldsymbol{v} \neq 0$. Then

$$T\boldsymbol{v} = \lambda\boldsymbol{v}, \quad \boldsymbol{v} \neq \mathbf{0}$$

and

$$\langle T\boldsymbol{v}, \boldsymbol{v} \rangle = \langle \lambda\boldsymbol{v}, \boldsymbol{v} \rangle = \lambda \langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$$

Since $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$, $\lambda \geq 0$.

"$\Leftarrow$": Since T is self-adjoint, by the Spectral Theorem, there exists an orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n\}$ of V whose basis vectors are eigenvectors of V:

$$\begin{aligned} T\boldsymbol{e}_1 &= \lambda_1 \boldsymbol{e}_1 \\ T\boldsymbol{e}_2 &= \lambda_2 \boldsymbol{e}_2 \\ &\vdots \\ T\boldsymbol{e}_n &= \lambda_n \boldsymbol{e}_n \end{aligned}$$

where $\lambda_i, i = 1, \dots, n$ are all the eigenvalues of T.

For any vector $\boldsymbol{v} \in V$,

$$\boldsymbol{v} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots c_n\boldsymbol{e}_n$$

Then

$$\begin{aligned} \langle T\boldsymbol{v}, \boldsymbol{v}\rangle &= \langle T(c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots c_n\boldsymbol{e}_n), c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots c_n\boldsymbol{e}_n\rangle \\ &= \langle c_1T\boldsymbol{e}_1 + c_2T\boldsymbol{e}_2 + \cdots c_nT\boldsymbol{e}_n, c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots c_n\boldsymbol{e}_n\rangle \\ &= \langle c_1\lambda_1\boldsymbol{e}_1 + c_2\lambda_2\boldsymbol{e}_2 + \cdots + c_n\lambda_n\boldsymbol{e}_n, c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + \cdots c_n\boldsymbol{e}_n\rangle \\ &= c_1^2\lambda_1 + c_2^2\lambda_2 + \cdots + c_n^2\lambda_n \end{aligned}$$

Since $\lambda_i \geq 0, i = 1, \dots, n$, the above quantity is nonnegative. So T is nonnegative.

Problem 6.

a. (6 points) Let $A \in \mathbb{F}^{n,n}$ be a square matrix that satisfies $A^2 = A$. Show that A is similar to the diagonal matrix.

$$C = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} = \begin{bmatrix} I_r & \\ & 0 \end{bmatrix}$$

That is, I_r is an identity square block of order r, $0 \leq r \leq n$.

Solution

Let $f(\lambda) = \lambda^2 - \lambda$. Then

$$f(A) = A^2 - A = 0$$

So the minimal polynomial of A divides f. So the eigenvalues can only be 0 or 1, and each Jordan block is of size 1×1. Rearranging the diagonal elements in the Jordan canonical form, we have A is similar to C.

b. (6 points) Let $A \in F^{n,n}$, $B \in F^{n,n}$ be square matrices such that $A^2 = A$, $B^2 = B$, and $AB = BA$. Suppose P_0 is an invertible matrix such that

$$P_0^{-1} A P_0 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Let $B_0 = P_0^{-1} B P_0$. Show that B_0 is in the form of

$$B_0 = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}$$

where B_1 is of order r, and $B_1^2 = B_1$ and $B_2^2 = B_2$.

Solution

Note

$$\begin{aligned} & AB = BA \Leftrightarrow \\ P_0 \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} P_0^{-1} \cdot P_0 B_0 P_0^{-1} &= P_0 B_0 P_0^{-1} \cdot P_0 \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} P_0^{-1} \end{aligned}$$

which is equivalent to

$$B_0 \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} = \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} B_0$$

So we can conclude that

$$B_0 = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}$$

In addition,

$$B_0^2 = (P_0^{-1}BP_0)(P_0^{-1}BP_0) = P_0^{-1}B^2P_0 = P_0^{-1}BP_0 = B_0$$

So

$$\begin{bmatrix} B_1^2 & \\ & B_2^2 \end{bmatrix} = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}$$

So $B_1^2 = B_1$ and $B_2^2 = B_2$.

c. (8 points) Let $A \in F^{n,n}$, $B \in F^{n,n}$ be square matrices such that $A^2 = A$, $B^2 = B$, and $AB = BA$. Show that there exists an invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal, and the diagonal entries are 0 and 1 for both. *(Hint: Let P_0 be the invertible matrix for A in part (b). Let Q_1 and Q_2 be invertible matrices that serve the same role for B_1 and B_2, respectively. Use P_0, Q_1 and Q_2 to construct the matrix P.)*

Solution

Since the B_1 and B_2 from (b) satisfy $B_1^2 = B_1$ and $B_2^2 = B_2$, there exist invertible matrices Q_1 and Q_2 such that

$$Q_1^{-1}B_1Q_1 = \begin{bmatrix} I_s & \\ & 0 \end{bmatrix}, s \leq r$$

and

$$Q_2^{-1}B_2Q_2 = \begin{bmatrix} I_t & \\ & 0 \end{bmatrix}, t \leq n - r$$

Let

$$Q = \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix}, \quad P = P_0Q$$

Then

$$\begin{aligned} P^{-1}BP &= P^{-1}(P_0B_0P_0^{-1})P = Q^{-1}P_0^{-1}(P_0B_0P_0^{-1})P_0Q = Q^{-1}B_0Q \\ &= \begin{bmatrix} Q_1^{-1} & \\ & Q_2^{-1} \end{bmatrix} \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix} \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} = \begin{bmatrix} Q_1^{-1}B_1Q_1 & \\ & Q_2^{-1}B_2Q_2 \end{bmatrix} \\ &= \begin{bmatrix} I_s & & & \\ & 0 & & \\ & & I_t & \\ & & & 0 \end{bmatrix} \end{aligned}$$

On the other hand

$$
\begin{aligned}
P^{-1}AP &= Q^{-1}P_0^{-1}AP_0Q \\
&= \begin{bmatrix} Q_1^{-1} & \\ & Q_2^{-2} \end{bmatrix} \begin{bmatrix} I_r & \\ & 0 \end{bmatrix} \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} = \begin{bmatrix} I_r & \\ & 0 \end{bmatrix}
\end{aligned}
$$

So we have found the matrix P.