University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 18, 2019

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
Steve Billups, Julien Langou (Chair), Yaning Liu.

Problem 1.

1. Let x and y be distinct eigenvectors of a matrix A such that $x + y$ is also an eigenvector of A. Is $x - y$ necessarily an eigenvector of A? Prove or give a counterexample.

2. Let S and T be linear operators on a finite-dimensional vector space over $\mathcal{C}$. Prove that ST and TS have the same eigenvalues.

Solution

1. Let λ, μ, and σ be the eigenvalues associated with x, y, and $x+y$, respectively. If x and y are linearly dependent, then $\lambda = \mu = \sigma$. If x and y are linearly independent, then $\lambda x + \mu y = \sigma(x+y)$, so $(\lambda - \sigma)x + (\mu - \sigma)y = 0$, which implies that $\lambda = \mu = \sigma$ (since x and y are linearly independent). In either case, the eigenvectors x, y and $x + y$ are associated with the same eigenvalue λ. Thus,

   $$A(x - y) = \lambda x - \lambda y = \lambda(x - y),$$

   . Furthermore, we note that x and y are distinct so that $x - y$ is nonzero, therefore $x - y$ is an eigenvector of A.

2. Let λ be an eigenvalue of ST associated with the eigenvector $u\,(u \neq 0)$. We have $(ST)u = \lambda u$. Multiplying on the left by T, it follows that $T(STu) = T(\lambda u)$. Rearranging, we get $(TS)(Tu) = \lambda(Tu)$.

   On the one hand, if $Tu \neq 0$, it follows that λ is an eigenvalue of TS associated with the eigenvector Tu. On the other hand, if $Tu = 0$, then λ is actually 0; we also have that, if $Tu = 0$, $\dim(\text{Null}(ST)) \neq 0$, then, using the fact that V is finite-dimensional, $\dim(\text{Null}(TS)) \neq 0$ (see fact 1 below). Thus, 0 is an eigenvalue of TS.

   We proved that if λ is an eigenvalue of ST, then it is an eigenvalue of TS. Reversing the roles of T and S, it can be shown that all eigenvalues of TS are also eigenvalues of ST. Thus, ST and TS have the same eigenvalues.

   <u>Fact 1:</u> Since the vector space V is finite-dimensional, we have that

   $$TS \text{ is invertible} \iff ST \text{ is invertible}$$

   Or in other words:

$$\dim(\text{Null}(TS)) \neq 0 \Longleftrightarrow \dim(\text{Null}(ST)) \neq 0$$

.

Proof: The fact that TS is invertible, implies that: (1a) TS is surjective; (1b) so, T is surjective; (1c) so, (this is where the finite dimensionality of V is needed,) T is invertible; (2a) TS is injective; (2b) so, S is injective; (2c) so, (this is where the finite dimensionality of V is important,) S is invertible. Since S and T are invertible, we have that ST is invertible.

Note: It is important to know that Fact 1 is not true in infinite dimension. It is also important to be able to give a counter-example.

Problem 2.

1. Prove or disprove: Two $n \times n$ real matrices with the same characteristic polynomials and the same minimal polynomials must be similar.

2. Let A be an $n \times n$ idempotent matrix (i.e., $A^2 = A$) with real entries. Prove that A must be diagonalizable.

Solution

1. Two matrices are similar if and only if they have the same Jordan form. Thus, the following two matrices (which are already in Jordan form) are not similar:

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   Observe that both matrices have chacteristic polynomial $(x-1)^4$ and minimal polynomial $(x-1)^2$. Thus, the statement is false.

2. Since $A^2 - A = 0$, the minimal polynomial for A must divide $x(x-1)$. Hence, the only possible eigenvalues are 0 and 1, and the factor corresponding to each eigenvalue appears to only the first power. Hence, in the Jordan form, all blocks are 1×1, so the Jordan form is diagonal. Thus, A is diagonalizable.

Problem 3. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be a list of n independent vectors in a vector space V. Show that the list of vectors

$$\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n + \boldsymbol{v}_1$$

is linearly independent if and only if n is odd.

Solution

Method 1: Using matrices

We work in the finite dimensional subspace $U = \text{Span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$. The matrix of $(\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n + \boldsymbol{v}_1)$ in the basis $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ of U is the n-by-n matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 \end{pmatrix}.$$

Now we know that:

1. The list of vectors $(\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n + \boldsymbol{v}_1)$ is linearly independent if and only if this matrix has rank n or equivallently if and only if the null space of this matrix has dimension 0 or equivallently if and only if the determinant of this matrix is not 0.
2. The list of vectors $(\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n + \boldsymbol{v}_1)$ is linearly dependent if and only if this matrix has rank less than n or equivallently if and only if the null space of this matrix has dimension greater than 0 or equivallently if and only if the determinant of this matrix is 0.

So, one can work on the matrix above to derive the answer to the question.

For example, if we perform the following sequence of row operations (that do not change the determinant),

$L_2 \leftarrow L_2 - L_1$, then $L_3 \leftarrow L_3 - L_2$, then $L_4 \leftarrow L_4 - L_3$, $\ldots$, then $L_{n-1} \leftarrow L_{n-1} - L_{n-2}$, then $L_n \leftarrow L_n - L_{n-1}$, we see that

$$\det(\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}) = \det(\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix})$$

$$= \det(\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}) = \det(\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix})$$

$$= \det(\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & (-1)^{n-3} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & (-1)^{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}) = \det(\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & (-1)^{n-3} \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 & (-1)^{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1+(-1)^{n-1} \end{pmatrix}).$$

So $\det(A) = 1 + (-1)^{n-1}$.

If n is even, then $\det(A) = 0$ and the list of vectors $(\boldsymbol{v}_1+\boldsymbol{v}_2, \boldsymbol{v}_2+\boldsymbol{v}_3, \dots, \boldsymbol{v}_{n-1}+\boldsymbol{v}_n, \boldsymbol{v}_n+\boldsymbol{v}_1)$ is linearly dependent

If n is odd, then $\det(A) = 2$ and the list of vectors $(\boldsymbol{v}_1+\boldsymbol{v}_2, \boldsymbol{v}_2+\boldsymbol{v}_3, \dots, \boldsymbol{v}_{n-1}+\boldsymbol{v}_n, \boldsymbol{v}_n+\boldsymbol{v}_1)$ is linearly independent

Method 2: More vector space-y method

$\boxed{\Rightarrow}$

(By contrapositive.) Assume n is even, then

$$(\boldsymbol{v}_1 + \boldsymbol{v}_2) - (\boldsymbol{v}_2 + \boldsymbol{v}_3) + \cdots + (\boldsymbol{v}_{n-1} + \boldsymbol{v}_n) - (\boldsymbol{v}_n + \boldsymbol{v}_1) = \mathbf{0}.$$

So the list of vectors is linearly dependent. (Note: We do not need to use the assumption that the list of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is linearly independent for this direction.)

$\boxed{\Leftarrow}$

Assume n is odd. Let $k_1, k_2, \ldots, k_n$ be scalars such that

$$k_1(\boldsymbol{v}_1 + \boldsymbol{v}_2) + k_2(\boldsymbol{v}_2 + \boldsymbol{v}_3) + \cdots + k_n(\boldsymbol{v}_n + \boldsymbol{v}_1) = \mathbf{0}$$

Then

$$(k_1 + k_n)\boldsymbol{v}_1 + (k_1 + k_2)\boldsymbol{v}_2 + \cdots + (k_{n-1} + k_n)\boldsymbol{v}_n = \mathbf{0}$$

Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is linearly independent, we have

$$k_n + k_1 = 0 \tag{1}$$
$$k_1 + k_2 = 0 \tag{2}$$
$$k_2 + k_3 = 0 \tag{3}$$
$$\vdots$$
$$k_{n-2} + k_{n-1} = 0 \tag{4}$$
$$k_{n-1} + k_n = 0 \tag{5}$$

On the one hand, Equation 1) gives $k_1 = -k_n$.

On the other hand, we have that $k_1 = -k_2$ (using Equation 2), therefore $k_1 = k_3$ (using Equation 3), etc. We see that, for all i from 2 to n, $k_1 = (-1)^{i-1}k_i$. In particular for $i = n$, we get $k_1 = (-1)^{n-1}k_n$.

So we have $k_1 = -k_n$ and $k_1 = (-1)^{n-1}k_n$. In the case n is odd, we see that this gives $k_1 = -k_n$ and $k_1 = k_n$, which means $k_1 = k_n = 0$. Since $k_1 = 0$, we get that

$$k_1 = k_2 = k_3 = \ldots = k_n = 0.$$

So the list of vectors $(\boldsymbol{v}_1 + \boldsymbol{v}_2) - (\boldsymbol{v}_2 + \boldsymbol{v}_3) + \cdots + (\boldsymbol{v}_{n-1} + \boldsymbol{v}_n) - (\boldsymbol{v}_n + \boldsymbol{v}_1)$ is linearly independent.

Problem 4.

1. Let V_1 and V_2 be two non-trivial (neither $\{\mathbf{0}\}$ nor V) subspaces of a vector space V on $\mathbb{F}$. Show that there exists vector $\boldsymbol{v} \in V$ such that $\boldsymbol{v} \notin V_1$ and $\boldsymbol{v} \notin V_2$.
2. Show the result holds for any s non-trivial subspaces. In other words, let $V_1, V_2, \ldots, V_s$ be s non-trivial subspaces of a vector space V on $\mathbb{F}$. Show that there exists vector $\boldsymbol{v} \in V$ such that $\boldsymbol{v} \notin V_1, \boldsymbol{v} \notin V_2, \ldots, \boldsymbol{v} \notin V_s$.

Solution

1. Since V_1 is a nontrivial subspace of V, there exists $\boldsymbol{w}_1 \notin V_1$. If in addition $\boldsymbol{w}_1 \notin V_2$, we are done. If $\boldsymbol{w}_1 \in V_2$, since V_2 is nontrivial, there exists vector $\boldsymbol{w}_2 \notin V_2$. If in addition $\boldsymbol{w}_2 \notin V_1$, we are done. Otherwise, now we have

$$\boldsymbol{w}_1 \notin V_1, \boldsymbol{w}_2 \in V_1, \boldsymbol{w}_1 \in V_2, \boldsymbol{w}_2 \notin V_2$$

   Letting $\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2$, we know that

$$\boldsymbol{v} \notin V_1, \boldsymbol{v} \notin V_2.$$

2. We use mathematical induction on s. When $s = 2$, we already showed the result holds. Suppose the result holds for $s - 1$, i.e., there exists $\boldsymbol{w} \in V$ such that $\boldsymbol{w} \notin V_i, i = 1, 2, \ldots, s-1$. For the subspace V_s, if $\boldsymbol{w} \notin V_s$, we are done. Otherwise, since V_s is nontrivial, there exists $\boldsymbol{w}_0 \in V$ such that $\boldsymbol{w}_0 \notin V_s$. So for any scalar k, the vector $k\boldsymbol{w} + \boldsymbol{w}_0 \notin V_s$. Note that for different scalars k_1 and k_2, the vectors $k_1\boldsymbol{w} + \boldsymbol{w}_0$ and $k_2\boldsymbol{w} + \boldsymbol{w}_0$ are not in the same $V_i, i = 1, 2, \ldots, s-1$. Otherwise,

$$(k_1\boldsymbol{w} + \boldsymbol{w}_0) - (k_2\boldsymbol{w} + \boldsymbol{w}_0) = (k_1 - k_2)\boldsymbol{w} \in V_i$$

   and so $\boldsymbol{w} \in V_i$.

   Now we can take s different scalars $k_1, k_2, \ldots, k_s$ then there is at least one vector in the following s vectors

$$k_1\boldsymbol{w} + \boldsymbol{w}_0, k_2\boldsymbol{w} + \boldsymbol{w}_0, \ldots, k_{s-1}\boldsymbol{w} + \boldsymbol{w}_0, k_s\boldsymbol{w} + \boldsymbol{w}_0$$

   that is not in any $V_1, V_2, \ldots, V_{s-1}$. Such a vector is what we want.

Problem 5. Find real numbers x, y and z such that

$$\int_0^1 \left(\ln(t) - x - yt - zt^2\right)^2 dt$$

is minimal.

Hints:

$$\int_0^1 \ln(t)dt = -1; \quad \int_0^1 t\ln(t)dt = -\frac{1}{4}; \quad \int_0^1 t^2\ln(t)dt = -\frac{1}{9};$$
$$\int_0^1 dt = 1; \quad \int_0^1 t\ dt = \frac{1}{2}; \quad \int_0^1 t^2\ dt = \frac{1}{3}; \quad \int_0^1 t^3\ dt = \frac{1}{4}; \quad \int_0^1 t^4\ dt = \frac{1}{5}.$$

Solution

We consider V the inner product space of the continuous square-integrable function on $(0,1]$. with the inner product:

$$\langle f, g\rangle = \int_0^1 f(t)\ g(t)\ dt.$$

We note that $\ln(t)$ is in V since $\ln(t)$ is square integrable on $(0,1]$.

If we call U the subspace $\mathrm{Span}(1,t,t^2)$ and y the vector $\ln(t)$, then we can rewrite our minimization problem as:

Find p in U such that $\|p-y\|$ is minimum.

To minimize this problem, we want to find p in U such that $p-y$, the residual, is orthogonal to the subspace U. So in other words, we want to find x, y and z such that

$$\begin{cases} \langle \ln(t) - (x+yt+zt^2), 1\rangle = 0 \\ \langle \ln(t) - (x+yt+zt^2), t\rangle = 0 \\ \langle \ln(t) - (x+yt+zt^2), t^2\rangle = 0 \end{cases}$$

$$\begin{cases} \int_0^1 \left(\ln(t) - (x+yt+zt^2)\right) dt = 0 \\ \int_0^1 \left(t\ln(t) - (xt+yt^2+zt^3)\right) dt = 0 \\ \int_0^1 \left(t^2\ln(t) - (xt^2+yt^3+zt^4)\right) dt = 0 \end{cases}$$

$$\begin{cases} \left(\int_0^1 \ln(t)dt\right) - x\left(\int_0^1 dt\right) - y\left(\int_0^1 t\ dt\right) - z\left(\int_0^1 t^2\ dt\right) = 0 \\ \left(\int_0^1 t\ln(t)dt\right) - x\left(\int_0^1 t\ dt\right) - y\left(\int_0^1 t^2\ dt\right) - z\left(\int_0^1 t^3\ dt\right) = 0 \\ \left(\int_0^1 t^2\ln(t)dt\right) - x\left(\int_0^1 t^2\ dt\right) - y\left(\int_0^1 t^3\ dt\right) - z\left(\int_0^1 t^4\ dt\right) = 0 \end{cases}$$

We have that

$$\int_0^1 \ln(t)dt = -1; \quad \int_0^1 t\ln(t)dt = -\frac{1}{4}; \quad \int_0^1 t^2\ln(t)dt = -\frac{1}{9};$$
$$\int_0^1 dt = 1; \quad \int_0^1 t\ dt = \frac{1}{2}; \quad \int_0^1 t^2\ dt = \frac{1}{3}; \quad \int_0^1 t^3\ dt = \frac{1}{4}; \quad \int_0^1 t^4\ dt = \frac{1}{5}.$$

So:

$$\begin{cases} -1 - x - \frac{1}{2}y - \frac{1}{3}z = 0 \\ -\frac{1}{4} - \frac{1}{2}x - \frac{1}{3}y - \frac{1}{4}z = 0 \\ -\frac{1}{9} - \frac{1}{3}x - \frac{1}{4}y - \frac{1}{5}z = 0 \end{cases}$$

So:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} 1 \\ \frac{1}{4} \\ \frac{1}{9} \end{pmatrix}.$$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{9} \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & \frac{1}{12} & \frac{1}{12} & -\frac{1}{4} \\ 0 & \frac{1}{12} & \frac{4}{45} & -\frac{2}{9} \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & \frac{16}{15} & -\frac{8}{3} \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & \frac{1}{15} & \frac{1}{3} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{3} & 1 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{10}{3} \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

So:

$$x = -\frac{10}{3}, \quad y = 8, \quad \text{and} \quad z = -5.$$

Note: We can also derive this system of equations by writing, right away, the normal equations:

$$\begin{pmatrix} \langle 1,1\rangle & \langle 1,t\rangle & \langle 1,t^2\rangle \\ \langle t,1\rangle & \langle t,t\rangle & \langle t,t^2\rangle \\ \langle t^2,1\rangle & \langle t^2,t\rangle & \langle t^2,t^2\rangle \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \langle 1,\ln(t)\rangle \\ \langle t,\ln(t)\rangle \\ \langle t^2,\ln(t)\rangle \end{pmatrix}$$

This gives exactly the same linear system of equations and method as previously.

Another solution

We perform Gram-Schmidt on $(1, t, t^2)$. Let $a_0 = 1$, $a_1 = t$ and $a_2 = t^2$, then

$$
\begin{aligned}
&w_0 = a_0 = 1. \\
&\|w_0\|^2 = \langle w_0, w_0 \rangle = \int_0^1 dt = 1. \\
&\|w_0\| = 1. \\
&q_0 = 1.
\end{aligned}
$$

$$
\begin{aligned}
&r_{01} = \langle a_1, q_0 \rangle = \int_0^1 t dt = \frac{1}{2}. \\
&w_1 = a_1 - q_0 r_{01} = t - \frac{1}{2}. \\
&\|w_1\|^2 = \int_0^1 (t - \frac{1}{2})^2 dt = \frac{1}{12}. \\
&\|w_1\| = \frac{1}{2\sqrt{3}}. \\
&q_1 = w_1 / \|w_1\| = \sqrt{3}\,(2t - 1)\,.
\end{aligned}
$$

$$
\begin{aligned}
&r_{02} = \langle a_2, q_0 \rangle = \int_0^1 t^2 dt = \frac{1}{3}. \\
&r_{12} = \langle a_2, q_1 \rangle = \int_0^1 \sqrt{3}\,(2t-1)\,t^2 dt = \frac{\sqrt{3}}{6}. \\
&w_2 = a_2 - q_0 r_{02} - q_1 r_{12} = t^2 - \frac{1}{3} - \frac{1}{2}\,(2t-1) = t^2 - t + \frac{1}{6}. \\
&\|w_2\|^2 = \int_0^1 (t^2 - t + \frac{1}{6})^2 dt = \frac{1}{180}. \\
&\|w_2\| = \int_0^1 (t^2 - t + \frac{1}{6})^2 dt = \frac{1}{6\sqrt{5}}. \\
&q_2 = w_2 / \|w_2\| = \sqrt{5}\left(6t^2 - 6t + 1\right).
\end{aligned}
$$

At this point, we have constructed an orthonormal basis q_0, q_1 and q_2 of the subspace U. (Where $U = \text{Span}(1, t, t^2)$.) (Orthonormal with respect to the inner product $\langle f, g \rangle = \int_0^1 f(t)\ g(t)\ dt$.) And, to repeat, q_0, q_1 and q_2 are:

$$
\begin{aligned}
&q_0 = 1. \\
&q_1 = \sqrt{3}\,(2t - 1)\,. \\
&q_2 = \sqrt{5}\left(6t^2 - 6t + 1\right).
\end{aligned}
$$

We call $x = \ln(t)$, and we now compute:

$$\langle x, q_0\rangle = \int_0^1 (\ln(t))dt = -1.$$

$$\langle x, q_1\rangle = \sqrt{3}\int_0^1 (2t\ln(t) - \ln(t))\,dt = \frac{\sqrt{3}}{2}.$$

$$\langle x, q_2\rangle = \sqrt{5}\int_0^1 \left(6t^2\ln(t) - 6t\ln(t) + \ln(t)\right)dt = -\frac{\sqrt{5}}{6}.$$

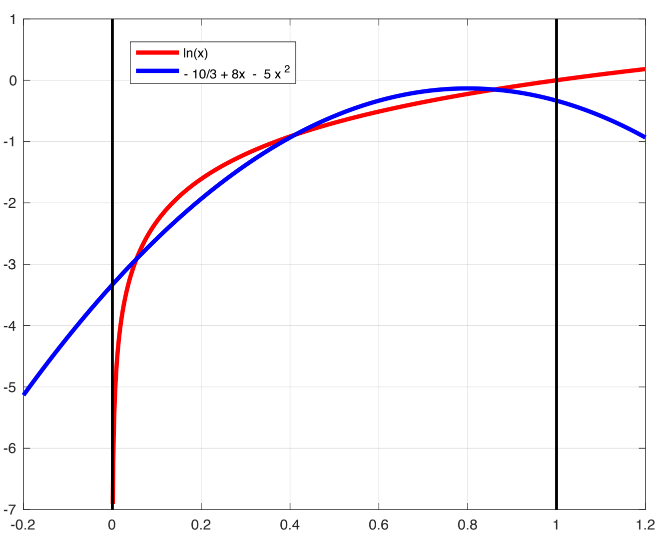
We are now ready to compute Px, the orthogonal projection of x onto U:

$$\begin{aligned} Px &= q_0\langle x, q_0\rangle + q_1\langle x, q_1\rangle + q_2\langle x, q_2\rangle, \\ &= (-1)\,(1) + \left(\frac{\sqrt{3}}{2}\right)\sqrt{3}\,(2t-1) - \left(\frac{\sqrt{5}}{6}\right)\sqrt{5}\left(6t^2 - 6t + 1\right), \\ &= -1 + 3t - \frac{3}{2} - 5t^2 + 5t - \frac{5}{6}, \\ &= -1 - \frac{3}{2} - \frac{5}{6} + 8t - 5t^2, \\ &= -\frac{10}{3} + 8t - 5t^2 \end{aligned}$$

So:

$$x = -\frac{10}{3}, \quad y = 8, \quad \text{and} \quad z = -5.$$

Interpretation



Out of all polynomials of degree 2, the polynomial $-\frac{10}{3} + 8t - 5t^2$ is the closest to $\ln(t)$ the sense that it minimizes the square of the area in between the two curves in between 0 and 1.

Problem 6.

Let n be an integer. Find all n-by-n matrices A with complex entries such that $A = A^H$ and

$$A^3 = 2A + 4I.$$

Solution

1. A has real eigenvalues since A is Hermitian ($A = A^H$).
2. The eigenvalues of A are roots of $x^3 - 2x - 4 = 0$.
3. We have that $x^3 - 2x - 4 = (x-2)(x^2 + 2x + 2)$. The polynomial $x^2 + 2x + 2$ does not have any real roots. So the only real root of $x^3 - 2x - 4$ is 2.
4. So A has only one eigenvalue and it is 2.
5. A is Hermitian ($A = A^H$) so A is diagonalizable. A has only one eigenvalue and this eigenvalue is 2. There is only one matrix that is diagonalizable and has spectrum $\{2\}$, this is $A = 2I$.
6. In conlusion, the only n-by-n matrix A such that $A = A^H$ and $A^3 - 2A - 4I = 0$ is

$$A = 2I.$$

Note:

- In item (5.), we state that "there is only one matrix that is diagonalizable and has spectrum $\{2\}$, this is $A = 2I$." Indeed, since A is diagonalizable, there exists an invertible matrix V and a diagonal matrix D such that $A = VDV^{-1}$. V represents a basis of eigenvectors for A and D are the associated eigenvalues. Since A has only one eigenvalue and this eigenvalue is 2, then $D = 2I$. So $A = VDV^{-1} = V(2I)V^{-1} = 2VV^{-1} = 2I$.. We proved that $A = 2I$. (There are other ways to prove this. For example, one can prove that A is an homothetie of ratio 2 by taking any vector v in $\mathbb{C}^n$ and proving that $Av = 2v$.)
- The statement would also be true if we replaced the assumption "A is Hermitian ($A = A^H$)" by "A only has real eigenvalues. (I.e., no complex eigenvalues.)" The fact that $A = 2I$ can be deduced by the facts that (1) $A^2+2A+2I$ must be invertible (otherwise A would have complex eigenvalues), and (2) so that $A^3 - 2A - 4I = 0$ implies $(A - 2I)(A^2 + 2A + 2I) = 0$ implies $A - 2I = 0$ implies $A = 2I$. In the

proof above, we use the fact that A Hermitian implies A diagonalizable, and then we use the fact that A is diagonalizable, but this is not necessary.

Problem 7. Let m be an integer. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m$ and $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_m$ be two lists of vectors in a real inner-product vector space V. Prove if

$$\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j \rangle = \left\langle \boldsymbol{\beta}_i, \boldsymbol{\beta}_j \right\rangle, \ i, j = 1, \ldots, m$$

then the subspaces $V_1 = \text{span}(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m)$ and $V_2 = \text{span}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m)$ are isomorphic.

Solution

Method 1: Prove that the dimensions of the subspaces V_1 and V_2 are the equal, and so V_1 and V_2 are isomorphic.

Students should be able to recognize the normal equations (also called the Gram matrix) when seeing:

$$\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j \rangle = \left\langle \boldsymbol{\beta}_i, \boldsymbol{\beta}_j \right\rangle, \ i, j = 1, \ldots, m$$

This problem is closely related to the standard result: "Let A be an n-by-m real matrix, prove that $\text{Rank}(A^T A) = \text{Rank}(A)$." In other words: "The rank of the matrix of the inner products (also called the normal equations or the Gram matrix) is the rank of the list of the vectors." In this question, we are given that the Gram matrices for the list of vectors $(\alpha_i)_i$ and the list of vectors $(\beta_i)_i$ are equal, so the Gram matrices have the same rank, so the list of vectors $(\alpha_i)_i$ and the list of vectors $(\beta_i)_i$ have the same rank, so their spans (V_1 and V_2) have the same dimension, so their spans (V_1 and V_2) are isomorphic. This is the right idea. However the dimension of the inner-product vector space might be infinite, so it is not possible to use a n-by-m matrix A. ("n is infinite.") And the inner product is not $x^T y$, but it is $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$. Another difficulty is that the classic proof of $\text{Rank}(A^T A) = \text{Rank}(A)$ in general goes by proving that the Null spaces are the same, therefore they have the same dimension, therefore the Rank are the same. This uses the rank theorem which is not true in infinite dimension.

After these observations, here is one way to go at this proof.

Let n_1 be the dimension of the subspace V_1. Let n_2 be the dimension of the subspace V_2. (Clearly $n_1 \leq m$ and $n_2 \leq m$.)

We will prove that $n_1 \leq n_2$. Since the list $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m)$ spans V_1, we can extract a basis of V_1 from this list of vector. Let $(\boldsymbol{\alpha}_{i_1}, \ldots, \boldsymbol{\alpha}_{i_{n_1}})$ be such a basis of V_1. We will prove that the list $(\boldsymbol{\beta}_{i_1}, \ldots, \boldsymbol{\beta}_{i_{n_1}})$ is linearly independent. (This will prove that $n_1 \leq n_2$.) Let $x_{i_1}, \ldots, x_{i_{n_1}}$ be scalars such that

$$x_{i_1} \boldsymbol{\beta}_{i_1} + \cdots + x_{i_{n_1}} \boldsymbol{\beta}_{i_{n_1}} = 0.$$

Let j be an integer between 1 and n_1, then

$$\left\langle \boldsymbol{\beta}_{i_j}, x_{i_1} \boldsymbol{\beta}_{i_1} + \cdots + x_{i_{n_1}} \boldsymbol{\beta}_{i_{n_1}} \right\rangle = 0.$$

$$x_{i_1}\left\langle \boldsymbol{\beta}_{i_j}, \boldsymbol{\beta}_{i_1}\right\rangle + \cdots + x_{i_{n_1}}\left\langle \boldsymbol{\beta}_{i_j}, \boldsymbol{\beta}_{i_{n_1}}\right\rangle = 0.$$

Now we use our main assumption $\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j \rangle = \langle \boldsymbol{\beta}_i, \boldsymbol{\beta}_j \rangle,\ i, j = 1, \ldots, m$ to write:

$$x_{i_1}\left\langle \boldsymbol{\alpha}_{i_j}, \boldsymbol{\alpha}_{i_1}\right\rangle + \cdots + x_{i_{n_1}}\left\langle \boldsymbol{\alpha}_{i_j}, \boldsymbol{\alpha}_{i_{n_1}}\right\rangle = 0.$$

$$\left\langle \boldsymbol{\alpha}_{i_j}, x_{i_1}\boldsymbol{\alpha}_{i_1} + \cdots + x_{i_{n_1}}\boldsymbol{\alpha}_{i_{n_1}}\right\rangle = 0.$$

Since the list of vectors $(\boldsymbol{\alpha}_{i_1}, \ldots, \boldsymbol{\alpha}_{i_{n_1}})$ spans V_1 (since it is a basis of V_1), we see that the vector $x_{i_1}\boldsymbol{\alpha}_{i_1} + \cdots + x_{i_{n_1}}\boldsymbol{\alpha}_{i_{n_1}}$ is orthogonal to all vectors in V_1. Also, this vector is in V_1. So this vector is the zero vector. This writes:

$$x_{i_1}\boldsymbol{\alpha}_{i_1} + \cdots + x_{i_{n_1}}\boldsymbol{\alpha}_{i_{n_1}} = 0.$$

Now the list of vectors $(\boldsymbol{\alpha}_{i_1}, \ldots, \boldsymbol{\alpha}_{i_{n_1}})$ is linearly independent (since it is a basis of V_1), so we must have that

$$x_{i_1} = \cdots = x_{i_{n_1}} = 0.$$

This proves that the list $(\boldsymbol{\beta}_{i_1}, \ldots, \boldsymbol{\beta}_{i_{n_1}})$ is linearly independent. This proves that $n_1 \leq n_2$.

By switching $\boldsymbol{\alpha}$'s with $\boldsymbol{\beta}$'s, V_1 with V_2, we can similarly prove that $n_2 \leq n_1$.

Consequently $n_1 = n_2$.

Consequently V_1 and V_2 have the same dimension.

Consequently V_1 and V_2 are isomorphic.

<u>Method 2: Construct an explicit map.</u>

Consider the map $\varphi : V_1 \to V_2$, defined by

$$\varphi(k_1\boldsymbol{\alpha}_1 + \cdots + k_m\boldsymbol{\alpha}_m) = k_1\boldsymbol{\beta}_1 + \cdots + k_m\boldsymbol{\beta}_m$$

First we show the map is well defined. Suppose

$$k_1\boldsymbol{\alpha}_1 + \cdots + k_m\boldsymbol{\alpha}_m = l_1\boldsymbol{\alpha}_1 + \cdots + l_m\boldsymbol{\alpha}_m \tag{6}$$

Then we have

$$(k_1 - l_1)\boldsymbol{\alpha}_1 + \cdots + (k_m - l_m)\boldsymbol{\alpha}_m = \mathbf{0}.$$

So it follows that

$$\begin{aligned}
&\langle (k_1 - l_1)\boldsymbol{\beta}_1 + \cdots + (k_m - l_m)\boldsymbol{\beta}_m, (k_1 - l_1)\boldsymbol{\beta}_1 + \cdots + (k_m - l_m)\boldsymbol{\beta}_m \rangle \\
&= \langle (k_1 - l_1)\boldsymbol{\alpha}_1 + \cdots + (k_m - l_m)\boldsymbol{\alpha}_m, (k_1 - l_1)\boldsymbol{\alpha}_1 + \cdots + (k_m - l_m)\boldsymbol{\alpha}_m \rangle \\
&= \langle \mathbf{0}, \mathbf{0} \rangle = 0
\end{aligned}$$

So

$$(k_1 - l_1)\boldsymbol{\beta}_1 + \cdots + (k_m - l_m)\boldsymbol{\beta}_m = \mathbf{0}$$

and

$$k_1\boldsymbol{\beta}_1 + \cdots + k_m\boldsymbol{\beta}_m = l_1\boldsymbol{\beta}_1 + \cdots + l_m\boldsymbol{\beta}_m \tag{7}$$

So the map φ is well defined. Now we show that the map is linear. Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in V_1$ and

$$\begin{aligned}
\alpha &= k_1\boldsymbol{\alpha}_1 + \cdots + k_m\boldsymbol{\alpha}_m \\
\beta &= l_1\boldsymbol{\alpha}_1 + \cdots + l_m\boldsymbol{\alpha}_m
\end{aligned}$$

$$\begin{aligned}
\varphi(\boldsymbol{\alpha} + \boldsymbol{\beta}) &= \varphi((k_1 + l_1)\boldsymbol{\alpha}_1 + \cdots + (k_m + l_m)\boldsymbol{\alpha}_m) \\
&= (k_1 + l_1)\boldsymbol{\beta}_1 + \cdots + (k_m + l_m)\boldsymbol{\beta}_m \\
&= (k_1\boldsymbol{\beta}_1 + \cdots + k_m\boldsymbol{\beta}_m) + (l_1\boldsymbol{\beta}_1 + \cdots + l_m\boldsymbol{\beta}_m) \\
&= \varphi(\boldsymbol{\alpha}) + \varphi(\boldsymbol{\beta})
\end{aligned}$$

and

$$\begin{aligned}
\varphi(\lambda\boldsymbol{\alpha}) &= \varphi(\lambda(k_1\boldsymbol{\alpha}_1 + \cdots + k_m\boldsymbol{\alpha}_m)) \\
&= \varphi(\lambda k_1\boldsymbol{\alpha}_1 + \cdots + \lambda k_m\boldsymbol{\alpha}_m)) \\
&= \lambda k_1\boldsymbol{\beta}_1 + \cdots + \lambda k_m\boldsymbol{\beta}_m \\
&= \lambda\varphi(\boldsymbol{\alpha})
\end{aligned}$$

Now we show φ is surjective and injective. First for surjectivity, for any $\boldsymbol{\beta} \in V_2$, $\boldsymbol{\beta} = k_1\boldsymbol{\beta}_1 + \cdots + k_m\boldsymbol{\beta}_m$, $k_1\boldsymbol{\alpha}_1 + \cdots + k_m\boldsymbol{\alpha}_m \in V_1$ is the corresponding inverse image in V_1. On the other hand, Eqn. (7) leads to Eqn. (6) by the same argument above, which means φ is injective. So φ is an isomorphism between V_1 and V_2, and V_1 and V_2 are isomorphic.

Problem 8. Let A and B be $m \times n$ and $n \times p$ matrices over $\mathbb{R}$, respectively.

1. Prove that $\dim(\mathrm{Null}(AB)) \leq \dim(\mathrm{Null}(A)) + \dim(\mathrm{Null}(B))$. (Hint: it may be convenient to let $V = \{x \in \mathbb{R}^p : ABx = 0\}$ and $W = \{y = Bx : x \in \mathbb{R}^p, Ay = 0\}$. Then consider the map $T_B : V \to W$ defined by $T_B : x \mapsto Bx$ for all $x \in V$).

2. Prove that $\mathrm{rank}(A) + \mathrm{rank}(B) \leq \mathrm{rank}(AB) + n$.

Solution

1. Let V, W and T_B be as defined in the hint. Observe that V is the null space of AB, so $\dim(V) = \dim(\mathrm{Null}(AB))$. Also, $W \subset \mathrm{Null}(A)$, so $\dim(W) \leq \dim(\mathrm{Null}(A))$. Moreover, $\mathrm{Null}(T_B) \subset \mathrm{Null}(B)$, so $\dim(\mathrm{Null}(T_B)) \leq \dim(\mathrm{Null}(B))$. (We note that we actually have $\mathrm{Null}(T_B) = \mathrm{Null}(B)$, so $\dim(\mathrm{Null}(T_B)) = \dim(\mathrm{Null}(B))$, but we only need the inequality for this proof.) Since for any linear transformation, the dimension of its domain equals the dimension of its image plus the dimension of its nullspace, we have

$$\begin{aligned}\dim(\mathrm{Null}(AB)) = \dim(V) &= \dim(\mathrm{Range}(T_B)) + \dim(\mathrm{Null}(T_B)) \\ &\leq \dim(W) + \dim(\mathrm{Null}(B)) \\ &\leq \dim(\mathrm{Null}(A)) + \dim(\mathrm{Null}(B)).\end{aligned}$$

2. The rank of a matrix is equal to the number of columns minus the dimension of the null space of the matrix. Thus, $\mathrm{rank}(A) = n - \dim(\mathrm{Null}(A))$, $\mathrm{rank}(B) = p - \dim(\mathrm{Null}(B))$ and $\mathrm{rank}(AB) = p - \dim(\mathrm{Null}(AB))$. Hence,

$$\begin{aligned}\mathrm{rank}(A) + \mathrm{rank}(B) &= n - \dim(\mathrm{Null}(A)) + p - \dim(\mathrm{Null}(B)) \\ &= n + p - \dim(\mathrm{Null}(A)) - \dim(\mathrm{Null}(B)) \\ &\leq n + p - \dim(\mathrm{Null}(AB)) \quad \text{from part (1)} \\ &= n + \mathrm{rank}(AB)\end{aligned}$$