University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 18, 2019

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. Submit as many solutions as you can. All solutions will be graded and your final grade will be based on your six best solutions.
- Each problem is worth 20 points.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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Problem 1.

1. Let x and y be distinct eigenvectors of a matrix A such that $x + y$ is also an eigenvector of A. Is $x - y$ necessarily an eigenvector of A? Prove or give a counterexample.

2. Let S and T be linear operators on a finite-dimensional vector space over $\mathcal{C}$. Prove that ST and TS have the same eigenvalues.

Problem 2.

1. Prove or disprove: Two $n \times n$ real matrices with the same characteristic polynomials and the same minimal polynomials must be similar.
2. Let A be an $n \times n$ idempotent matrix (i.e., $A^2 = A$) with real entries. Prove that A must be diagonalizable.

Problem 3. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be a list of n independent vectors in a vector space V. Show that the list of vectors

$$\boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_2 + \boldsymbol{v}_3, \ldots, \boldsymbol{v}_{n-1} + \boldsymbol{v}_n, \boldsymbol{v}_n + \boldsymbol{v}_1$$

is linearly independent if and only if n is odd.

Problem 4.

1. Let V_1 and V_2 be two non-trivial (neither $\{\mathbf{0}\}$ nor V) subspaces of a vector space V on $\mathbb{F}$. Show that there exists vector $\boldsymbol{v} \in V$ such that $\boldsymbol{v} \notin V_1$ and $\boldsymbol{v} \notin V_2$.

2. Show the result holds for any s non-trivial subspaces. In other words, let $V_1, V_2, \ldots, V_s$ be s non-trivial subspaces of a vector space V on $\mathbb{F}$. Show that there exists vector $\boldsymbol{v} \in V$ such that $\boldsymbol{v} \notin V_1, \boldsymbol{v} \notin V_2, \ldots, \boldsymbol{v} \notin V_s$.

Problem 5. Find real numbers x, y and z such that

$$\int_0^1 \left(\ln(t) - x - yt - zt^2\right)^2 dt$$

is minimal.

Hints:

$$\int_0^1 \ln(t)dt = -1; \quad \int_0^1 t\ln(t)dt = -\frac{1}{4}; \quad \int_0^1 t^2\ln(t)dt = -\frac{1}{9};$$
$$\int_0^1 dt = 1; \quad \int_0^1 t\ dt = \frac{1}{2}; \quad \int_0^1 t^2\ dt = \frac{1}{3}; \quad \int_0^1 t^3\ dt = \frac{1}{4}; \quad \int_0^1 t^4\ dt = \frac{1}{5}.$$

Problem 6.

Let n be an integer. Find all n-by-n matrices A with complex entries such that $A = A^H$ and

$$A^3 = 2A + 4I.$$

Problem 7. Let m be an integer. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \ldots, \boldsymbol{\alpha}_m$ and $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \ldots, \boldsymbol{\beta}_m$ be two lists of vectors in a real inner-product vector space V. Prove if

$$\langle \boldsymbol{\alpha}_i, \boldsymbol{\alpha}_j \rangle = \langle \boldsymbol{\beta}_i, \boldsymbol{\beta}_j \rangle, \; i, j = 1, \ldots, m$$

then the subspaces $V_1 = \mathrm{span}(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_m)$ and $V_2 = \mathrm{span}(\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m)$ are isomorphic.

Problem 8. Let A and B be $m \times n$ and $n \times p$ matrices over $\mathbb{R}$, respectively.

1. Prove that $\dim(\text{Null}(AB)) \leq \dim(\text{Null}(A)) + \dim(\text{Null}(B))$. (Hint: it may be convenient to let $V = \{x \in \mathbb{R}^p : ABx = 0\}$ and $W = \{y = Bx : x \in \mathbb{R}^p, Ay = 0\}$. Then consider the map $T_B : V \to W$ defined by $T_B : x \mapsto Bx$ for all $x \in V$).

2. Prove that $\text{rank}(A) + \text{rank}(B) \leq \text{rank}(AB) + n$.