University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
May 25th, 2018

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. All solutions will be graded and your final grade will be based on your *six best problems*. Your final score will be out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce a complete proof.
- Parts of a multipart question are not necessarily worth the same number of points.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any questions.

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

Let $V = \mathbb{P}^2(\mathbb{R})$, the space of real-valued polynomials of total degree less than or equal to 2. Let $S = \{x, 1 - x, 1 - x^2\}$ be a set of vectors in V.

(a) Show that S is a basis of V.

(b) Let $T : V \to V$ given by $p(x) \to xp'(x)$. Find the matrix of T with respect to S.

(c) Is T invertible? If not, express its nullspace in terms of S.

Problem 2

Suppose $V_1, V_2, \ldots, V_s$ are subspaces of V.

(a) Show that the sum of $V_1, V_2, \ldots, V_s$ is a direct sum if and only if

$$V_i \cap \sum_{j \neq i} V_j = \mathbf{0}, i = 1, \ldots, s.$$

(b) Show that the sum of $V_1, V_2, \ldots, V_s$ is a direct sum if and only if

$$V_1 \cap V_2 = \mathbf{0}, (V_1 + V_2) \cap V_3 = \mathbf{0}, \ldots, (V_1 + V_2 + \cdots + V_{s-1}) \cap V_s = \mathbf{0}.$$

Problem 3

Let A be a real matrix satisfying $A^3 = A$.

(a) Prove that A can be diagonalized.

(b) If A is a 3×3 matrix, how many different possible similarity classes are there for A?

Problem 4

Let T be a self-adjoint operator on an n-dimensional inner product space V. Let λ_0 be an eigenvalue of T. Show that the (algebraic) multiplicity of λ_0 equals dim $E(\lambda_0, T)$, i.e., the dimension of the eigenspace of T corresponding to λ_0.

Problem 5

Let T and S be linear maps on inner product space V. For any vector $\boldsymbol{v} \in V$,

$$\langle T\boldsymbol{v}, T\boldsymbol{v}\rangle = \langle S\boldsymbol{v}, S\boldsymbol{v}\rangle .$$

Show that range T and range S are isomorphic.

Problem 6

Let V be a complex-valued vector space.

(a) Give an example of an operator T that is surjective but not invertible.

(b) Given an example of an operator S that has no eigenvalue.

(c) Let $V = \mathbb{C}^\infty$. Let U be the set of vectors in V with finitely many non-zero entries. Prove that U is an infinite-dimensional subspace of V.

Problem 7

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Problem 8

Let W_1 and W_2 be two subspaces of an n-dimensional vector space V, and $\dim W_1 + \dim W_2 = n$. Show that there exists an operator T on V such that

$$\text{null } T = W_1 \quad \text{and} \quad \text{range } T = W_2.$$