University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 12, 2018

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. You are asked to submit solutions to *6 problems*. If you submit solutions to more than six problems, you must indicate which problems to grade. If you do not indicate which problems to grade, only the first six solutions will contribute to your grade. Your final score will be out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce a complete proof.
- Parts of a multipart question are not necessarily worth the same number of points.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any questions.

1.	________	5.	________
2.	________	6.	________
3.	________	7.	________
4.	________	8.	________
		Total	________

DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = (1, 0, 1, 0),\ v_2 = (0, 1, 0, 1),\ v_3 = (1, -1, 2, 1),\ v_4 = (-1, 0, 2, 0), \quad \text{and } w = (3, 1, 0, 1).$$

(a) Prove that $B = \{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$, and give the coordinates of w with respect to this basis.

(b) Determine for which v_i the set $B_i = (B \setminus \{v_i\}) \cup \{w\}$ is a basis of $\mathbb{R}^4$. Justify your answer for each v_i.

Problem 2[1]

Let V be a finite vector space over the reals. Let $m \geq 2$, and let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be a collection of vectors in V where $\alpha_m \neq \mathbf{0}$. Prove that the vectors

$$\beta_1 = \alpha_1 + k_1\alpha_m, \quad \beta_2 = \alpha_2 + k_2\alpha_m, \quad \cdots, \quad \beta_{m-1} = \alpha_{m-1} + k_{m-1}\alpha_m$$

are linearly independent for any numbers $k_1, k_2, \ldots, k_{m-1}$ if and only if the vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent.

[1] On the administered exam, there was a mistake in the statement of this problem such that the quantification of $k_1, \ldots, k_{m-1}$ occurred outside the biconditional. In that case, the linear independence of the β_is does not imply the linear independence of the α_js, as can be seen by the example $\alpha_1 = (0, -1, 0)$, $\alpha_2 = (-1, 0, 0)$, $\alpha_3 = (1, 1, 0)$, and $k_1 = k_2 = 1$. This mistake was taken into account when grading the submitted exams.

Problem 3

Let $V = \mathbb{P}_n(\mathbb{R})$, the real valued polynomials of total degree less than or equal to n, and let the inner product on V be given by $\int_a^b p_1(x)p_2(x)\ dx$ for real numbers a and b satisfying $a < b$.
Let $o_n(x)$ denote the orthogonal polynomial of degree n resulting from applying the Gram-Schmidt process to the monomial basis $\{1, x, x^2, \ldots, x^n\}$.

(a) Prove that $o_n(x)$ is orthogonal to any polynomial of degree $n-1$ or less.

(b) Prove that all roots of $o_n(x)$ are real.

(c) Prove that $o_n(x)$ has a root in $[a, b]$.

Problem 4

Let D be in $\mathbb{R}^{n \times n}$ and diagonal with entries $d_1 < d_2 < \ldots < d_n$. Let Z be a symmetric rank 1 matrix with non-zero eigenvalue ρ and no zero entries. Prove that if λ is an eigenvalue of $D + Z$ and v is a corresponding eigenvector, then

(a) $Zv \neq 0$.

(b) D and $D + Z$ do not have any common eigenvalues.

Problem 5

Let $A, B, C, D \in F^{n\times n}$ be square matrices.

(a) Prove that $T(X) = AXB + CX + XD$ is a linear transformation on $F^{n\times n}$;

(b) If $C = D = \mathbf{0}$, prove that T is invertible if and only if A and B are invertible.

(c) Let $A = B = C = D$, and equip $F^{n\times n}$ with the matrix 2-norm. Prove a non-trivial upper bound on the operator norm of T in terms of the singular values of A. You do not have to prove that $\|A\|_2$ is a submultiplicative norm.

Problem 6

Let $A, B \in \mathbb{R}^{n \times n}$. Prove the following statements.

(a) If A is similar to B (in notation: $A \sim B$), then for any natural number k and real number c, the following hold:

$$A^k \sim B^k, \qquad cA \sim cB.$$

(b) If A is similar to B, then for any polynomial $f(x)$, $f(A) \sim f(B)$.

(c) If A is invertible, AB and BA are similar.

(d) Suppose matrix C is obtained by interchanging the i, j rows, and i, j columns of A. Then C is similar to A.

Problem 7

Let V be an arbitrary inner product space over the reals.

(a) Show that there must exist vectors $\alpha_1, \beta_1 \in V$, $\alpha_1 \neq \beta_1$, such that

$$(\alpha_1, \beta_1) > 0,$$

where (a, b) denotes the inner product of a and b. Also show that there must exist vectors $\alpha_2, \beta_2 \in V$, $\alpha_2 \neq \beta_2$, such that

$$(\alpha_2, \beta_2) < 0.$$

(b) Let M be the set $\{[\alpha, \beta] : (\alpha, \beta) > 0\}$, i.e., the set of all the pairs of vectors from V whose inner product is positive, and let N be the set $\{[\alpha, \beta] : (\alpha, \beta) < 0\}$. Show that there exists a bijection between M and N.

Problem 8

Let A be an $n \times n$ matrix with real entries. Prove that

(a) all the eigenvalues of A are zero if and only if there exists a positive integer m such that $A^m = \mathbf{0}$;

(b) if $A^m = 0$, then the determinant of $A + I$ equals 1, where I is the $n \times n$ identity matrix.