# University of Colorado Denver
# Department of Mathematical and Statistical Sciences
# Applied Linear Algebra Ph.D. Preliminary Exam
# January 12, 2018

Name: ______________________________________________

**Exam Rules:**

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. You are asked to submit solutions to *6 problems*. If you submit solutions to more than six problems, you must indicate which problems to grade. If you do not indicate which problems to grade, only the first six solutions will contribute to your grade. Your final score will be out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . . ).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce a complete proof.
- Parts of a multipart question are not necessarily worth the same number of points.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any questions.

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**Applied Linear Algebra Preliminary Exam Committee:**
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## Problem 1

Consider the following vectors in $\mathbb{R}^4$:

$$v_1 = (1,0,1,0),\ v_2 = (0,1,0,1),\ v_3 = (1,-1,2,1),\ v_4 = (-1,0,2,0), \quad \text{and } w = (3,1,0,1).$$

(a) Prove that $B = \{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$, and give the coordinates of $w$ with respect to this basis.

(b) Determine for which $v_i$ the set $B_i = (B \setminus \{v_i\}) \cup \{w\}$ is a basis of $\mathbb{R}^4$. Justify your answer for each $v_i$.

**Solution:**

(a) Let $A$ be the $4 \times 4$ matrix $A$ whose $i$th row is $v_i$. The reduced row echelon form of $A$ is the identity $I$, and hence $A$ has full rank. Thus, the rows of $A$ are linearly independent, and since the size of $B$ matches the dimension of the space, $B$ is a basis.

To find the coordinates of $w$ with respect to the basis $B$, we solve $A^T x = w$, finding that $x = (2,1,0,-1)$.

(b) From the coordinates for $w$ with respect to $B$, we note that $2v_1 + 1v_2 - 3v_4 = w$. Hence, for $v_i$ in $\{v_1, v_2, v_4\}$ can be written in terms of $w$ and the elements of $B \setminus \{v_i\}$. Since $v_i$ is in the span of $(B \setminus \{v_i\}) \cup \{w\}$ and the span of $B$ is $\mathbb{R}^4$, the span of $(B \setminus \{v_i\}) \cup \{w\}$ is $\mathbb{R}^4$. Since $(B \setminus \{v_i\}) \cup \{w\}$ has four elements in $\mathbb{R}^4$, they must also form a basis.

For $B_3 = (B \setminus \{v_3\}) \cup \{w\}$, note that by the above relation $B_3$ is not a linearly independent set of vectors. Hence, $B_3$ cannot be a basis for $\mathbb{R}^4$.

## Problem 2[1]

Let $V$ be a finite vector space over the reals. Let $m \geq 2$, and let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be a collection of vectors in $V$ where $\alpha_m \neq \mathbf{0}$. Prove that the vectors

$$\beta_1 = \alpha_1 + k_1\alpha_m, \quad \beta_2 = \alpha_2 + k_2\alpha_m, \quad \cdots, \quad \beta_{m-1} = \alpha_{m-1} + k_{m-1}\alpha_m$$

are linearly independent for any numbers $k_1, k_2, \ldots, k_{m-1}$ if and only if the vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent.

**Solution:** Suppose for any numbers $k_1, k_2, \ldots, k_{m-1}$, $\beta_1, \beta_2, \ldots, \beta_{m-1}$ are linearly independent. If $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_m$ are linearly dependent, then $\alpha_m$ can be expressed as a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}$. So there exists $c_1, c_2, \ldots, c_{m-1}$, such that

$$\alpha_m = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_{m-1}\alpha_{m-1}.$$

Since $\alpha_m \neq 0$, $c_1, c_2, \ldots, c_{m-1}$ cannot be all zero. WLOG, suppose $c_1 \neq 0$. Then if

$$k_1 = -\frac{1}{c_1}, \quad , k_2 = \cdots = k_{m-1} = 0,$$

$\beta_1, \beta_2, \ldots, \beta_{m-1}$ are linearly dependent, since $\beta_1 = \alpha_1 - \frac{1}{c_1}\alpha_m$ is a linear combination of $\alpha_2, \ldots, \alpha_{m-1}$, and thus a linear combination of $\beta_2, \ldots, \beta_{m-1}$, which contradicts to the assumption. So $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_m$ must be linearly independent.
On the other hand, suppose $\alpha_1, \alpha_2, \ldots, \alpha_{m-1}, \alpha_m$ are linearly independent. Now we prove $\beta_1, \beta_2, \ldots, \beta_{m-1}$ are linearly independent. Suppose $l_1, l_2, \ldots, l_{m-1}$ exist such that

$$l_1\beta_1 + l_2\beta_2 + \cdots + l_{m-1}\beta_{m-1} = 0$$

Substituting $\beta_i = \alpha_i + k_i\alpha_m, i = 1, 2, \ldots, m-1$, then

$$l_1\alpha_1 + l_2\alpha_2 + \cdots + l_{m-1}\alpha_{m-1} + (l_1k_1 + \cdots + l_{m-1}k_{m-1})\alpha_m = 0$$

Since $\alpha_1, \alpha_2, \ldots, \alpha_m$ are linearly independent,

$$l_1 = l_2 = \cdots = l_{m-1} = 0,$$
$$l_1k_1 + \cdots + l_{m-1}k_{m-1} = 0.$$

So $\beta_1, \beta_2, \ldots, \beta_{m-1}$ are linearly independent.

[1] On the administered exam, there was a mistake in the statement of this problem such that the quantification of $k_1, \ldots, k_{m-1}$ occurred outside the biconditional. In that case, the linear independence of the $\beta_i$s does not imply the linear independence of the $\alpha_j$s, as can be seen by the example $\alpha_1 = (0, -1, 0)$, $\alpha_2 = (-1, 0, 0)$, $\alpha_3 = (1, 1, 0)$, and $k_1 = k_2 = 1$. This mistake was taken into account when grading the submitted exams.

## Problem 3

Let $V = \mathbb{P}_n(\mathbb{R})$, the real valued polynomials of total degree less than or equal to $n$, and let the inner product on $V$ be given by $\int_a^b p_1(x)p_2(x)\ dx$ for real numbers $a$ and $b$ satisfying $a < b$.

Let $o_n(x)$ denote the orthogonal polynomial of degree $n$ resulting from applying the Gram-Schmidt process to the monomial basis $\{1, x, x^2, \ldots, x^n\}$.

(a) Prove that $o_n(x)$ is orthogonal to any polynomial of degree $n-1$ or less.

(b) Prove that all roots of $o_n(x)$ are real.

(c) Prove that $o_n(x)$ has a root in $[a, b]$.

**Solution:**

(a) By the Gram-Schmidt process, $o_n(x)$ is orthogonal to $o_i(x)$ $i < n$, and $\text{span}(o_1, o_2, \ldots, o_{n-1}) = \text{span}(1, x, \ldots, x^{n-1})$. Thus $x^i$, $i < n$, equals $\sum_{j=1}^{n-1} \alpha_j o_j$. $(o_n, x^i) = \sum_j \alpha_j (o_n, o_j) = 0$.

(b) Let $o_n(x)$ have a complex root $z$. By the Fundamental Theorem of Algebra, it must also have a root $\bar{z}$, and hence it must have a quadratic factor $q(x) = (x - (z + \bar{z})x + |z|^2)$ which does not change sign on $\mathbb{R}$. Let $o_n(x) = q(x)p(x)$. By part (a), $(o_n(x), p(x)) = 0$ as $p(x)$ has smaller degree. But then we have $(p(x)q(x), p(x)) = (p^2(x), q(x)) \neq 0$, a contradiction.

(c) Actually you can prove than $o_n(x)$ has $n$ distinct roots in $[a, b]$; to prove the weaker statement here, assume that $c$ is a root of $o_n(x)$ not in $[a, b]$. Then $(x - c)$ is a factor of $o_n(x)$ with quotient $p(x)$. Again, $(o_n(x), p(x)) = 0$, and $(x - c)$ does not change sign on $[a, b]$. Again, we have $0 = (p^2(x), (x - c)) \neq 0$, a contradiction.

## Problem 4

Let $D$ be in $\mathbb{R}^{n\times n}$ and diagonal with entries $d_1 < d_2 < \ldots < d_n$. Let $Z$ be a symmetric rank 1 matrix with non-zero eigenvalue $\rho$ and no zero entries. Prove that if $\lambda$ is an eigenvalue of $D+Z$ and $v$ is a corresponding eigenvector, then

(a) $Zv \neq 0$.

(b) $D$ and $D+Z$ do not have any common eigenvalues.

**Solution:** Let $\lambda$ also be an eigenvalue of $D$; then $\lambda = d_i$. As $(D+Z-\lambda I)v = 0$, $(e_i, D+Z-\lambda I)v) = 0$. Splitting the inner product we have: $(e_i,(D-\lambda I)v)+(e_i, Zv) = 0$. $Z$ admits an eigendecomposition $\rho z z^\top$, $z_j \neq 0 \forall j$. Plugging in we have

$$\begin{aligned}(e_i,(D-\lambda I)v) + \rho e_i^\top z(z^\top v) &= 0\\ (D-\lambda I)e_i, v) + \rho e_i^\top z(z^\top v) &= 0\\ 0 + \rho z_i(z^\top v) &= 0 \Rightarrow (z,v) = 0.\end{aligned}$$

**(ii) first:** If $(z,v) = 0$ then $\rho z z^\top v = 0 \rightarrow Zv = 0$. Hence $(D+\lambda I)v = 0$, i.e. $v$ is an eigenvector of $D$. But $D$ has distinct eigenvalues, so hence $v$ is in span$\{e_i\}$. But then we have $0 = z^\top v = z^\top k e_i (k \neq 0) = kz_i \neq 0$. Hence $D$ and $D+Z$ do not have any common eigenvalues.

(i): from (ii), $(D-\lambda I)$ is invertible; then $(D-\lambda I)v \neq 0$ as $v \neq 0$. As $(D-\lambda I)v = Zv$, $Zv \neq 0$.

## Problem 5

Let $A, B, C, D \in F^{n\times n}$ be square matrices.

(a) Prove that $T(X) = AXB + CX + XD$ is a linear transformation on $F^{n\times n}$;

(b) If $C = D = \mathbf{0}$, prove that $T$ is invertible if and only if $A$ and $B$ are invertible.

(c) Let $A = B = C = D$, and equip $F^{n\times n}$ with the matrix 2-norm. Prove a non-trivial upper bound on the operator norm of $T$ in terms of the singular values of $A$. You do not have to prove that $\|A\|_2$ is a submultiplicative norm.

**Solution:**

(a) Let $X, Y \in F^{n\times n}$.

$$\begin{aligned} T(X+Y) &= A(X+Y)B + C(X+Y) + (X+Y)D \\ &= (AXB + CX + XD) + (AYB + CY + YD) \\ &= T(X) + T(Y) \end{aligned}$$

and

$$\begin{aligned} T(kX) &= A(kX)B + C(kX) + (kX)D \\ &= k(AXB + CX + XD) = kT(X) \end{aligned}$$

So $T$ is a linear transformation on $F^{n\times n}$.

(b) When $C = D = \mathbf{0}$, supposing $A$ and $B$ are invertible, for any $H \in F^{n\times n}$,

$$T(A^{-1}HB^{-1}) = A(A^{-1}HB^{-1})B = H,$$

which means $T$ is surjective. In addition, when $X \neq Y$, $T(X) = AXB \neq AYB = T(Y)$, which means $T$ is injective. So $T$ is invertible.

On the other hand, if $T$ is invertible, then $A$ and $B$ must be invertible matrices. Otherwise, supposing $A$ is not invertible, there exists a non-zero square matrix $H$ such that $AH = \mathbf{0}$. So

$$T(H) = AHB = \mathbf{0}B = \mathbf{0},$$

which contradicts the fact that $T$ is invertible.

(c) The operator norm of $T$ is equal to

$$\begin{aligned} \max_{\|X\|=1} \|T(X)\| = \max_{\|X\|=1} \|AXA + AX + XA\| &\leq \|A\|\|X\|\|A\| + \|A\|\|X\| + \|X\|\|A\| \\ &= \|A\|^2 + 2\|A\| \\ &= \sigma_1^2 + 2\sigma_1, \end{aligned}$$

where $\sigma_1 = \|A\|$, the largest singular value of $A$.

## Problem 6

Let $A, B \in \mathbb{R}^{n\times n}$. Prove the following statements.

(a) If $A$ is similar to $B$ (in notation: $A \sim B$), then for any natural number $k$ and real number $c$, the following hold:

$$A^k \sim B^k, \qquad cA \sim cB.$$

(b) If $A$ is similar to $B$, then for any polynomial $f(x)$, $f(A) \sim f(B)$.

(c) If $A$ is invertible, $AB$ and $BA$ are similar.

(d) Suppose matrix $C$ is obtained by interchanging the $i, j$ rows, and $i, j$ columns of $A$. Then $C$ is similar to $A$.

**Solution:** Since $A \sim B$, there exists an invertible square matrix $P$ such that

$$P^{-1}AP = B$$

(a) If $k = 0$, $A^k = B^k = I$, so $A^k \sim B^k$. If $k > 0$, we have

$$(P^{-1}AP)^k = P^{-1}A^kP = B^k$$

So $A^k \sim B^k$. Also, $P^{-1}(cA)P = cP^{-1}AP = cB$, so $cA \sim cB$.

(b) let $f(x) = c_0x^m + c_1x^{m-1} + \cdots + c_{m-1}x + c_m$. Then

$$\begin{aligned} f(B) &= c_0(P^{-1}AP)^m + c_1(P^{-1}AP)^{m-1} + \cdots + c_{m-1}(P^{-1}AP) + c_mI \\ &= c_0P^{-1}A^mP + c_1P^{-1}A^{m-1}P + \cdots + c_{m-1}P^{-1}AP + c_mP^{-1}IP \\ &= P^{-1}\left(c_0A^m + c_1A^{m-1} + \cdots + c_{m-1}A + c_mI\right)P \\ &= P^{-1}f(A)P \end{aligned}$$

so $f(A) \sim f(B)$.

(c) Since $A$ is invertible, $A^{-1}$ exists, and

$$BA = (A^{-1}A)\cdot(BA) = A^{-1}(AB)A$$

So $AB \sim BA$.

(d) Interchanging rows $i$ and $j$ of matrix $A$ is equivalent to left-multiplying $A$ by $P(i, j)$, where $P(i, j)$ is obtained by interchanging the $i, j$ rows of the identity matrix. Interchanging columns $i, j$ of matrix $A$ is equivalent to right-multiplying $A$ by the same matrix $P(i, j)$. So

$$C = P(i, j)AP(i, j)$$

Since $P(i, j)$ is invertible and $P(i, j)^2 = I$,

$$P(i, j)^{-1} = P(i, j).$$

So

$$C = P(i, j)^{-1}AP(i, j),$$

i.e., $C$ is similar to $A$.

## Problem 7

Let $V$ be an arbitrary inner product space over the reals.

(a) Show that there must exist vectors $\alpha_1, \beta_1 \in V$, $\alpha_1 \neq \beta_1$, such that

$$(\alpha_1, \beta_1) > 0,$$

where $(a, b)$ denotes the inner product of $a$ and $b$. Also show that there must exist vectors $\alpha_2, \beta_2 \in V$, $\alpha_2 \neq \beta_2$, such that

$$(\alpha_2, \beta_2) < 0.$$

(b) Let $M$ be the set $\{[\alpha, \beta] : (\alpha, \beta) > 0\}$, i.e., the set of all the pairs of vectors from $V$ whose inner product is positive, and let $N$ be the set $\{[\alpha, \beta] : (\alpha, \beta) < 0\}$. Show that there exists a bijection between $M$ and $N$.

**Solution:**

(a) Taking any non-zero vector $\alpha \in V$ and positive real number $k \neq 1$, then

$$(\alpha, k\alpha) = k(\alpha, \alpha) > 0,$$

i.e., there exist $\alpha \neq \beta$ such that $(\alpha, \beta) > 0$.

In addition, $\alpha \neq -\alpha$, and $(\alpha, -\alpha) = -(\alpha, \alpha) < 0$.

(b) Let $\varphi : [\alpha, \beta] \to [\alpha, -\beta]$, i.e.,

$$\varphi([\alpha, \beta]) = [\alpha, -\beta]$$

If $[\alpha, \beta] \neq [\alpha_1, \beta_1]$, i.e., $\alpha \neq \alpha_1$, or/and $\beta \neq \beta_1$, then $\alpha \neq \alpha_1$ or/and $-\beta \neq -\beta_1$. So

$$[\alpha, -\beta] \neq [\alpha_1, -\beta_1],$$

which means $\varphi$ is a injection from $M$ to $N$.

On the other hand, suppose $[\alpha, \gamma] \in N$, i.e. $(\alpha, \gamma) < 0$. Then $(\alpha, -\gamma) > 0$, and so $[\alpha, -\gamma] \in M$, and $\varphi([\alpha, -\gamma]) = [\alpha, \gamma]$,

i.e., $\varphi$ is a surjection from $M$ to $N$. So $\varphi$ is a bijection from $M$ to $N$.

## Problem 8

Let $A$ be an $n \times n$ matrix with real entries. Prove that

(a) all the eigenvalues of $A$ are zero if and only if there exists a positive integer $m$ such that $A^m = \mathbf{0}$;

(b) if $A^m = 0$, then the determinant of $A+I$ equals 1, where $I$ is the $n \times n$ identity matrix.

**Solution:**

(a) Let the Jordan canonical form of $A$ be

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}.$$

Then there exists an invertible matrix $P$ such that $P^{-1}AP = J$. If $A^m = \mathbf{0}$, then

$$J^m = \begin{bmatrix} J_1^m & & & \\ & J_2^m & & \\ & & \ddots & \\ & & & J_s^m \end{bmatrix} = P^{-1}A^mP = \mathbf{0}, \qquad J_i^m = \begin{bmatrix} \lambda_i^m & & & \\ & \lambda_i^m & & \\ & * & \ddots & \\ & & & \lambda_i^m \end{bmatrix}$$

By $\lambda_i^m = 0, i = 1, \dots, s$, we have $\lambda_i = 0, i = 1, \dots, s$.

On the other hand, if all the eigenvalue of $A$ are zero, then the Jordan canonical form of $A$, $J$, must be of the form

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}$$

where

$$J_i = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

Taking $m \geq \max \text{order}(J_i)$, then $J_i^m = \mathbf{0}$, which means $J^m = 0$, and thus $A^m = 0$.

(b) If $A^m = 0$, then all the eigenvalues of $A$ are zero, by (1). Let the Jordan canonical form of $A$ be

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}.$$

So the diagonal elements of each Jordan block are all zero. So the main diagonal elements of $J$ are all zero, and $J + I$ is a lower-triangular matrix with 1's on the main diagonal. So

$$\det(A + I) = \det(P^{-1}JP + I) = \det(P^{-1}(J + I)P) = \det(J + I) = 1.$$