University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Linear Algebra Ph.D. Preliminary Exam
January 13, 2017

Name: ____

Exam Rules:

- This exam lasts 4 hours.
- There are 8 problems. Each problem is worth 20 points. You are asked to submit solutions to *6 problems*. If you submit solutions to more than six problems, you must indicate which problems to grade. If you do not indicate which problems to grade, only the first six solutions will contribute to your grade. Your final score will be out of 120 points.
- You are not allowed to use books or any other auxiliary material on this exam.
- Start each problem on a separate sheet of paper, write only on one side, and label all of your pages in consecutive order (*e.g.*, use 1-1, 1-2, 1-3, . . . , 2-1, 2-2, 2-3, . . .).
- Read all problems carefully, and write your solutions legibly using a dark pencil or pen in "essay-style" using full sentences and correct mathematical notation.
- Justify your solutions: cite theorems you use, provide counterexamples for disproof, give clear but concise explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, you may not merely quote or rephrase that theorem as your solution; instead, you must produce a complete proof.
- If you feel that any problem or any part of a problem is ambiguous or may have been stated incorrectly, please indicate your interpretation of that problem as part of your solution. Your interpretation should be such that the problem is not trivial.
- Please ask the proctor if you have any other questions.

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Applied Linear Algebra Preliminary Exam Committee:
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Problem 1

(a) Consider the set $V = \mathbb{R}^2$ with the usual addition $+$ and the "scalar multiplication" $* : \mathbb{R} \times V \to V$ given by

$$a * (x, y)^T := (a \cdot x, |a| \cdot y)^T \qquad \text{for } a \in \mathbb{R}, \text{ and } (x, y)^T \in V.$$

Prove that V with $+$ (as addition) and $*$ as multiplication satisfies all conditions for a vector space but one. State which condition fails and give an example.

(b) Let V and W be vector spaces over $\mathbb{C}$ and let $f : V \to W$ be linear. Prove that if $U \subseteq V$ is a vector subspace, then the image

$$f(U) := \{f(u) : u \in U\} \subseteq W$$

also is a vector subspace.

Solution: (a) We consider all the vector space axioms:

- $(V, +)$ is an abelian group, it is the standard addition.
- The distributive law with respect to multiplication with a scalar is valid, too:

$$\begin{aligned} a * \big((x_1, y_1)^T + (x_2, y_2)^T\big) &= a * (x_1 + x_2, y_1 + y_2)^T = (a(x_1 + x_2), |a|(y_1 + y_2))^T \\ &= (ax_1 + ax_2, |a|y_1 + |a|y_2)^T = (ax_1, |a|y_1)^T + (ax_2, |a|y_2)^T \\ &= a * (x_1, y_1)^T + a * (x_2, y_2)^T. \end{aligned}$$

- The associative law with respect to multiplication with respect to several scalars holds, too: $(ab) * (x, y)^T = ((ab)x, |ab|y)^T = (a(bx), |a|(|b|y))^T = a * (bx, |b|y)^T = a * \big(b * (x, y)^T\big)$.
- Multiplication with scalar 1 is neutral: $1 * (x, y)^T = (1x, |1|y)^T = (x, y)^T$.
- However, the distributive law with respect to multiplication with a sum of two scalars is violated: $(a + b) * (x, y)^T = ((a + b)x, |a + b|y)^T$ and $a * (x, y)^T + b * (x, y)^T = \big((ax, |a|y)^T + (bx, |b|y)^T\big) = ((a + b)x, (|a| + |b|)y)^T$.

  The necessary equality of terms only holds for $|a+b| = |a|+|b|$, which in turn only holds for a and b that have the same sign. But for example $|-3 + 4| = 1 \neq 7 = |-3| + |4|$.

(b) We show that $f(U)$ satisfies all the conditions to be a vector subspace:

(i) As f is linear and $0 \in U$, we have $0 = f(0) \in f(U)$.

(ii) Let $w_1, w_2 \in f(U)$. Then there are $u_1, u_2 \in U$ such that $f(u_i) = w_i$, $i = 1, 2$. As U is a vector subspace, then also $u_1 + u_2 \in U$ and as f is linear, we obtain $w_1 + w_2 = f(u_1) + f(u_2) = f(u_1 + u_2) \in f(U)$.

(iii) Let $w \in f(U)$ and $\lambda \in K$, then there is a $u \in U$ such that $f(u) = w$. as U is a vector subspace, we obtain $\lambda u \in U$ and as f is linear then also $\lambda w = \lambda f(u) = f(\lambda u) \in f(U)$.

Problem 2

For each $a \in \mathbb{R}$ let the vector subspace $U_a \subseteq \mathbb{R}^4$ be the set of solutions for the linear equation

$$x_1 + x_2 + ax_4 = 0.$$

(a) For each $a, b \in \mathbb{R}$ with $a \neq b$, compute the intersection $U_a \cap U_b$ of the subspaces corresponding to a and b, and prove that the intersection does not depend on the choice of $a, b \in \mathbb{R}$ if $a \neq b$.

(b) For which $a, b \in \mathbb{R}$ is $U_a \cup U_b$ a subspace? Prove your claim and give explicit counterexamples for any negative cases.

Solution: (a) Let $a \neq b$. Then the intersection $U_a \cap U_b$ is the set of solutions of a system of linear equations with extended coefficient matrix

$$\begin{bmatrix} 1 & 1 & 0 & a & 0 \\ 1 & 1 & 0 & b & 0 \end{bmatrix}$$

We solve to:

$$\begin{bmatrix} 1 & 1 & 0 & a & 0 \\ 0 & 0 & 0 & a-b & 0 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 1 & 0 & a & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We subtracted the first row from the second one, then divided the second row by $a-b$ (which is nonzero), and finally subtracted the a-multiple of the second row from the first one. We obtain the set of solutions with free parameters $\lambda := x_2$ and $\mu := x_3$:

$$U_a \cap U_b = L = \left\{(-\lambda, \lambda, \mu, 0) \in \mathbb{R}^4 : \lambda, \mu \in \mathbb{R}\right\},$$

independently of the choice of a and b.

(b) If $a = b$, then $U_a = U_b$, and so $U_a \cup U_b$ is a subspace. Suppose that $a \neq b$. Let $v = (a, 0, 0, -1)$ and $w = (b, 0, 0, -1)$. Note that $v \in U_a$ and $w \in U_b$, so both v and w are in $U_a \cup U_b$. However, $v + w = (a + b, 0, 0, -2)$ is in neither U_a nor U_b (by direct calculation), and hence $v + w \notin U_a \cup U_b$. Thus $U_a \cup U_b$ is not closed under addition and hence is not a subspace.

Problem 3

Prove that the largest singular value of a linear transformation $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is equal to

$$\max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \frac{\langle y, Ax\rangle}{\|x\|\,\|y\|}.$$

Solution: We know that we can decompose A as $U\Sigma V^*$, U, V orthogonal, and Σ diagonal, containing the ordered singular values of A, $\sigma_1 \geq \sigma_2 \geq \ldots \sigma_n$. We also have $Av_i = \sigma_i u_i$, where u_i and v_i are in the orthonomal bases corresponding to U and V. Thus σ_1, the largest singular value, equals $\langle u_1, \sigma_1 u_1\rangle$, but that equals $\langle u_1, Av_1\rangle$. Hence σ_1 is less than

$$\max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \frac{\langle y, Ax\rangle}{\|x\|\,\|y\|}.$$

Assume that there exists unit vectors $x \neq v_1$ and $y \neq u_1$, such that $\langle y, Ax\rangle > \sigma_1$. Expanding y and x in the corresponding bases-then we have $\langle y, Ax\rangle = \langle \sum \alpha_i u_i, \sum \sigma_i \beta_i u_i\rangle$. By orthonormality, this reduces to $\sum_i \alpha_i\beta_i\sigma_i$, which by our assumption is greater than σ_1, hence $(\sigma_1)^{-1}\sum_i \alpha_i\beta_i\sigma_i > 1$. But $\|y\|^2 = \sum_i \alpha_i^2 = 1$, and $\|x\|^2 = \sum_i \beta_i^2$=1, and $(\sigma_1)^{-1}\sum_i \alpha_i\beta_i\sigma_i \leq \sum_i \alpha_i\beta_i \leq 1/2\sum_i \alpha_i^2 + \beta_i^2 = 1$ by Young's inequality, a contradiction. Thus $\sigma_1 \geq \max_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \frac{\langle y, Ax\rangle}{\|x\|\|y\|}$.

We may shorten the argument by recalling that $\|A\| = \sigma_1 = \max_x \frac{\|Ax\|}{\|x\|} = \sigma_1$, multiplying by $\|y\| / \|y\|$, and using Cauchy-Schwarz, to obtain $(\forall y)$

$$\sigma_1 \geq \max_x \frac{|\langle y, Ax\rangle|}{\|x\|\,\|y\|}$$

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Problem 4

(a) Let three bases A, B, C in $\mathbb{R}^3$ be given, as well as the basis transformation matrices

$$S_{A,B} = \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 0 \\ 0 & 2 & 7 \end{pmatrix} \quad \text{and} \quad S_{A,C} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

Compute the basis transformation matrix $S_{C,B}$.

Note: $S_{A,B}$ is the representation matrix of the identity $id : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto x$, where the coordinates of x with respect to B are given before the mapping, and the coordinates with respect to A are given after the mapping.

(b) Let $B = \{b_1, b_2, b_3\} \subset V := \mathbb{R}^3$ with $b_1 = (1,1,1)^T$, $b_2 = (2,0,-1)^T$, $b_3 = (2,3,1)^T$ be a basis of $\mathbb{R}^3$ and

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

i) Prove that $\varphi_A : \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$ is a vector space isomorphism.

ii) Compute a basis C of V, such that $D_{B,C}(\varphi) = I_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $D_{B,C}(\varphi)$ is the representation matrix of φ. ($D_{B,C}(\varphi)$ takes coordinates with respect to B before the mapping and gives coordinates with respect to C after the mapping.)

Solution: (a) We have $S_{A,B}S_{B,C} = S_{A,C}$: Let $x \in \mathbb{R}^3$ be a coordinate vector with respect to C, then $y = S_{B,C}x$ is the corresponding coordinate vector with respect to B. As $S_{A,B}y$ is the coordinate vector corresponding to y with respect to A, we obtain $S_{A,C}x = S_{A,B}y = S_{A,B}S_{B,C}x$. This holds for all $x \in \mathbb{R}^3$, and thus $S_{A,B}S_{B,C} = S_{A,C}$, respectively $S_{C,B} = S_{B,C}^{-1} = (S_{A,C})^{-1}S_{A,B}$. We obtain:

$$S_{C,B} = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 5 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 5 & 4 & 0 \\ 0 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & -4 & -12 \\ -5 & 4 & 22 \end{pmatrix}.$$

(b)

i) We only have to show that the rank of the square matrix A is 3. With a row reduction, we obtain

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 2 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

which shows $\operatorname{rang}(A) = 3$.

ii) Let $C = \{c_1, c_2, c_3\}$ be a basis such that

$$D_{B,C}(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain the following necessary conditions:

$$\begin{aligned} \varphi(b_1) &= 1 \cdot c_1 + 0 \cdot c_2 + 0 \cdot c_3, \\ \varphi(b_2) &= 0 \cdot c_1 + 1 \cdot c_2 + 0 \cdot c_3, \\ \varphi(b_3) &= 0 \cdot c_1 + 0 \cdot c_2 + 1 \cdot c_3, \end{aligned}$$

In short, $\varphi(b_i) = c_i$, $i = 1, 2, 3$. As φ is an isomorphism of vector spaces, it maps a basis to a basis. This implies that the resulting set C is a basis. We obtain $c_1 = (4, 3, 2)^T$,$c_2 = (-3, 2, 1)^T$ and $c_3 = (6, 3, 3)^T$.

Problem 5

(a) Let $A \in \mathbb{C}^{n\times n}$ be a square matrix and let $\tau \in \mathbb{C}$ be a scalar. For which $\mu \in \mathbb{C}$ is the following statement true? Prove your claim.

Let $\lambda \in \mathbb{C}$. Then λ is an eigenvalue of A if and only if μ is an eigenvalue of $A + \tau \cdot I_n$, where I_n is the $n \times n$ identity matrix.

(b) For a real number a, let

$$A = \begin{pmatrix} -a & a & 0 \\ 0 & 0 & 0 \\ 0 & -a & a \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

Compute A^{101}.

Solution: (a) Claim: λ is an eigenvalue of A if and only if $\mu = \lambda + \tau$ is an eigenvalue of $A + \tau \cdot I_n$.

Proof: Let $v \in K^n \setminus \{0\}$. Then v is an eigenvector for eigenvalue λ of $A \iff Av = \lambda v \iff (A + \tau \cdot I_n)v = (\lambda + \tau)v \iff v$ is an eigenvector for eigenvalue $\lambda + \tau$ of $A + \tau \cdot I_n$.
(b)
Note: The sample solution provides an answer for A, A^2, and A^3, while the problem only asked for A^2 and A^3.

For $a = 0$ clearly $A^{101} = A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So assume $a \neq 0$. Then $\chi_A = (x + a)x(x - a)$ has three distinct roots, so A has three distinct eigenvalues $-a, 0, a$. The corresponding eigenspaces can be read off directly: $E_{-a} = \langle (1,0,0)^T \rangle$, $E_0 = \langle (1,1,1)^T \rangle$ and $E_a = \langle (0,0,1)^T \rangle$.

With $S = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ we then have $\operatorname{diag}(-a, 0, a) = S^{-1}AS$ and thus

$$\begin{aligned} A^{101} &= S\operatorname{diag}(-a,0,a)^{101}S^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \operatorname{diag}(-a^{101}, 0, a^{101}) \cdot \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -a^{101} & a^{101} & 0 \\ 0 & 0 & 0 \\ 0 & -a^{101} & a^{101} \end{pmatrix}. \end{aligned}$$

Problem 6

a) Let $A \in \mathbb{C}^{n\times n}$ be a *nilpotent* matrix, which means that there is a $k \in \mathbb{N}$, such that $A^k = 0$. Compute all eigenvalues of A.

b) Let

$$A = \begin{pmatrix} 12 & 20 & -2 & -10 \\ -3 & -4 & 1 & -1 \\ 9 & 14 & -2 & -4 \\ 3 & 6 & 0 & -6 \end{pmatrix}.$$

Compute A^2 and A^3, all corresponding eigenvalues and their algebraic and geometric multiplicities. Further, state which eigenspaces (of both matrices) are subsets of each other.

Solution:

a) Claim: If λ is an eigenvalue of A and v a corresponding eigenvector of A, then λ^l is an eigenvalue of A^l for all $l \in \mathbb{N}$ and v a corresponding eigenvector.

Proof (only the induction step, the start is clear): $A^{l+1}v = A \cdot A^l v = A(\lambda^l v) = \lambda^l Av = \lambda^l \cdot \lambda v = \lambda^{l+1}v$.

Let now $A^k = 0$, then 0 is the only eigenvalue of A^k, and thus also of A.

b) Though the problem only asked for A^2 and A^3, we provide an answer for all of A, A^2, and A^3. Note that $A^2 = \begin{pmatrix} 36 & 72 & 0 & -72 \\ -18 & -36 & 0 & 36 \\ 36 & 72 & 0 & -72 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $A^3 = 0$. Using (a), we see that 0 is the only eigenvalue in all three cases. Further $E_0(A) \subset E_0(A^2) \subset E_0(A^3) = \mathbb{R}^4$. As all polynomials over $\mathbb{C}$ fully decompose into linear factors, we obtain $\chi_A(x) = \chi_{A^2}(x) = \chi_{A^3}(x) = x^4$, and the algebraic multiplicities always are 4. Further $E_0(A) = \ker(A) = \begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & 2 & 1 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $E_0(A^2) = \ker(A^2) = \begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Thus the geometric multiplicity goes from 2 to 3 to 4.

Problem 7

Let I be the identity operator on $\mathbb{C}^2$. Prove or disprove that I has infinitely many self-adjoint square roots.

Solution: Any orthogonal operator O will have $OO^* = I$. We now just need to select an orthogonal D such that O is also self-adjoint ($D = D^*$). Any reflection satisfies this. Pick a vector $v = [\cos\theta \ \sin\theta]^\top$. Then the projector P onto v is

$$= vv^\top = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}$$

and the corresponding (Householder) reflector is $I - 2P$, or

$$\begin{pmatrix} 1 - 2\cos^2\theta & -2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 1 - 2\sin^2\theta \end{pmatrix}.$$

We may rewrite this as

$$\begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}.$$

Alternatively, we may may write down D as

$$\begin{pmatrix} a & b + ci \\ b - ci & d \end{pmatrix},$$

which is Hermitian. D^2 equals

$$\begin{pmatrix} a^2 + b^2 + c^2 & a(b - ci) + d(b + ci) \\ a(b - ci) + d(b + ci) & b^2 + c^2 + d^2 \end{pmatrix}.$$

As $D^2 = I$, we get $a^2 - d^2 = 0$ (by subtracting the two constraints for the diagonal entries of I), i.e. $a = \pm d$. If $a = d$, $b = 0$ (as $2ab = 0$), while c is unconstrained, and thus any pair s.t. $a^2 + c^2 = 1$ is a square root. If $a = -d$, we get $c = 0$, and any pair $a^2 + b^2 = 1$ gives a (real) square root.

Problem 8

Let A be a real $n \times n$ matrix. A *real cube root* of A is a real $n \times n$ matrix B satisfying $B^3 = A$.

(a) Show that if A is symmetric then it has a real cube root.

(b) Find (with proof) a real 3×3 matrix which does not have a real cube root.

Solution: (a) Since A is real and symmetric, it has real eigenvalues and is diagonalizable. Let D be a diagonal matrix and P an invertible matrix such that $A = PDP^{-1}$. Let C be the diagonal matrix formed by taking the cube root of each entry in D, and let $B = PCP^{-1}$. Then

$$B^3 = (PCP^{-1})^3 = PC(P^{-1}P)C(P^{-1}P)CP^{-1} = PC^3P^{-1} = PDP^{-1} = A.$$

(b) *Solution 1.* Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and let B be a matrix with $B^3 = A$. If λ is an eigenvalue of B, then λ^3 is an eigenvalue of A, so $\lambda^3 = 0$ and therefore $\lambda = 0$. Thus the Jordan canonical form J of B is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{or } J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In all cases, $J^3 = 0$. Since J is the Jordan form for B, there exists an invertible matrix P such that $B = PJP^{-1}$. But then

$$B^3 = (PJP^{-1})^3 = PJ^3P^{-1} = P0P^{-1} = 0 \neq A,$$

giving a contradiction.

Solution 2. Let A be a 3×3 nilpotent matrix, where $A \neq 0$, $A^2 \neq 0$, and $A^3 = 0$. The matrix A given in Solution 1 is such a matrix. Suppose that $A = B^3$. Since $B^9 = (B^3)^3 = A^3 = 0$, which means the minimal polynomial for B divides λ^9. Hence the minimal polynomial of B is a power of λ, and by the Cayley–Hamilton Theorem has degree at most 3. However, $B^6 = (B^3)^2 = A^2 \neq 0$, which implies that the minimal polynomial cannot be λ, λ^2, nor λ^3. This is a contradiction.