Applied Analysis Prelim Exam
January 14, 2020

Name:

Take a deep breath. Relax. **Take your time to read each problem**. If you are confused or stuck on a problem, either ask a question or move on to another problem.

There are 8 total problems. Do all 4 problems in the first part (problems 1 to 4), and pick two problems in the second part (problems 5 to 8). Do not submit more than two solved problems from the second part; if you do, we will grade the the first two attempted with the lowest numbers. Do not submit multiple alternative solutions to any problem or its part; if you do, we will not try to pick the better solution or look for explanations in the alternative text, but grade only whichever solution comes first and stop.

Some problems have several parts, in which case your average score on all parts determines your score for that problem.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
Total:	120	

Part I (Do all 4 problems)

(1) Let (y_n) be a sequence of real numbers. Prove that there exists a continuous function $f : \Re \to \Re$ with $f(\frac{1}{n}) = y_n$ if and only if the sequence (y_n) converges.

(2) (a) Prove that if (c_n) is an increasing bounded sequence, then it converges.
(b) Let
$$c_n = \sum_{i=1}^{n} \frac{1}{n+i}$$
Prove that (c_n) converges. Hint: Use part (a).

(3) Prove that $\Re$ is not compact, using (a) open covers, and (b) sequences.

(4) Suppose $f_n : \Re \to \Re$ is bounded for each $n = 1, 2, \ldots$, i.e., $|f_n(x)| \leq c_n$. Suppose $f_n \to f$ uniformly. Prove or find a counterexample: f is bounded.

Part 2 (choose 2 out of 4 problems)

(5) (a) Derive the power series for $1/(1+x)$ and prove it converges uniformly on $[-r, r]$ for $0 < r < 1$.
(b) Integrate to get the power series for $\ln(1+x)$ and justify your steps.
(c) Find $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ from (b) and justify your steps.

(6) Let $\hat{\mathcal{C}}[0,1]$ be the set of continuous function, $f : [0,1] \to [0,1]$ For $f, g \in \hat{\mathcal{C}}[0,1]$, let

$$d_1(f,g) = \int_0^1 |f(x) - g(x)|dx$$
$$d_2(f,g) = \sup_{0 \le x \le 1} |f(x) - g(x)|.$$

Let X be the set of sequences $(x_1, x_2, \ldots)$ where each $x_i \in [0,1]$, and for $x, y \in X$, let

$$\hat{d}(x,y) = \sup_i |x_i - y_i|$$

Let $F : \hat{\mathcal{C}}[0,1] \to X$ be given by

$$F(f) = (f(1), f(\frac{1}{2}), f(\frac{1}{3}), \ldots)$$

(a) Verify that $d_1, d_2, \hat{d}$ are metrics.
(b) For $i = 1, 2$ prove or find a counterexample: $F : (\hat{\mathcal{C}}[0,1], d_i) \to (X, \hat{d})$ is continuous

(7) (a) Suppose $x > -1$. Use the Taylor's Theorem to express $f(x) = \ln(1+x)$ with the remainder in the x^3 term.
(b) Determine

$$\lim_{n \to \infty} e^{-nx}(1 + \frac{x}{n})^{n^2}$$

for any $x \in \Re$. Hint: Take the logarithm and use part (a).

(8) (a) Prove or find a counterexample: If f is continuous on $[a,b]$ then for any $\epsilon > 0$ there is a polynomial $p(x)$ such that $\int_a^b |f(x) - p(x)|dx < \epsilon$.
(b) Prove or find a counterexample: If f is Riemann integrable on $[a,b]$ then for any $\epsilon > 0$ there is a polynomial $p(x)$ such that $\int_a^b |f(x) - p(x)|dx < \epsilon$.