(1) Let \((X, d)\) be a metric space, \(K \subset X\) be nonempty, and let \(K'\) denote the set of limit points of \(K\).
Define the closure of \(K\) as \(\overline{K} := K \cup K'\). Prove that \(\overline{K}\) is both (1) closed, and (2) if \(F\) is closed and \(K \subset F\), then \(\overline{K} \subset F\). In other words, prove that \(\overline{K}\) is the smallest closed set containing \(K\).

Proof. We prove \(\overline{K}\) is closed by showing \(\overline{K}^c\) is open. Consider any \(x \in \overline{K}^c\), then \(x\) is neither in \(K\) nor a limit point of \(K\), so there exists a neighborhood around \(x\) that does not intersect \(K\). This implies that \(\overline{K}^c\) is open, which proves (1).

If \(F\) is closed then \(F\) contains all of its limit points, so if \(K \subset F\), then by definition \(K' \subset F\), which implies \(\overline{K} \subset F\). \qed
(2) Let \((X, d)\) be a metric space and \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(X\). Prove that if there exists \(x \in X\) such that for every subsequence \((x_{n_k})_{k \in \mathbb{N}}\) there exists a subsequence \((x_{n_{k_j}})_{j \in \mathbb{N}}\) such that \(x_{n_{k_j}} \to x\), then \(x_n \to x\).

**Proof.** Suppose that every subsequence \((x_{n_k})_{k \in \mathbb{N}}\) has a subsequence \((x_{n_{k_j}})_{j \in \mathbb{N}}\) such that \(x_{n_{k_j}} \to x\), but assume to the contrary that \(x_n \not\to x\). Then, there exists an \(\epsilon > 0\) such that for every \(N \in \mathbb{N}\) there exists \(n > N\) such that \(d(x_n, x) \geq \epsilon\). Choose such an \(\epsilon\) and construct a subsequence \((x_{n_k})_{k \in \mathbb{N}}\) such that \(d(x_{n_k}, x) \geq \epsilon\) inductively as follows. For \(k = 1\), let \(n_1\) be the first index of the sequence such that \(d(x_{n_1}, x) \geq \epsilon\). For \(k = 2\), let \(n_2 > n_1\) be an index such that \(d(x_{n_2}, x) \geq \epsilon\). Having chosen the first \(k\) terms in the subsequence, choose \(n_{k+1}\) such that \(n_{k+1} > n_k\) and \(d(x_{n_{k+1}}, x) \geq \epsilon\). By construction, any subsequence \((x_{n_{k_j}})_{j \in \mathbb{N}}\) of \((x_{n_k})_{k \in \mathbb{N}}\) will also have the property that \(d(x_{n_{k_j}}, x) \geq \epsilon\), which contradicts the hypothesis. \(\Box\)
(3) Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, \(K \subset X\) nonempty and open, and \(f : K \to Y\). Let \(\overline{K}\) denote the closure of \(K\) (see problem 1 for definition). Suppose \(Y\) is complete and \(f\) is uniformly continuous.

(a) (15 points) Prove that there exists a unique uniformly continuous function \(\overline{f} : \overline{K} \to Y\) such that \(\overline{f}(x) = f(x)\) for every \(x \in K\). We call \(\overline{f}\) the extension of \(f\) to \(\overline{K}\).

(b) (5 points) Give (1) an example showing the necessity of the condition that \(Y\) is complete, and (2) an example showing that even if \(Y\) is complete but \(f\) is only continuous, then there may not be an extension of \(f\) to \(\overline{K}\) that is continuous.

Part (a)

Proof. Consider any \(x \in \overline{K}\), then there exists \((x_n) \subset K\) such that \(x_n \to x\), which implies \((x_n)\) is Cauchy. Since \(f\) is uniformly continuous, \((f(x_n))\) is Cauchy by a standard result. Since \(Y\) is complete, \((f(x_n))\) converges to some number that we define to be \(\overline{f}(x)\).

This way of defining \(\overline{f}\) is both well-defined and unique since if \(x_n \to x\) and \(z_n \to x\), then we can define \((u_n)\) so that every even term defines the subsequence given by \((x_n)\) and every odd term defines the subsequence given by \((z_n)\), which is Cauchy by construction (this is easily proven by an \(\epsilon/2\) argument). Since \((u_n)\) is Cauchy with convergent subsequences, it converges by a standard result, and the uniqueness of limits immediately gives \(f(z_n) \to \overline{f}(x)\).

We now show that \(\overline{f}\) is uniformly continuous on \(\overline{K}\).

Let \(\epsilon > 0\).

Since \(f\) is uniformly continuous on \(K\), there exists \(\delta > 0\) such that \(d_Y(f(x), f(z)) < \epsilon/3\) for any \(x, z \in K\) with \(d_X(x, z) < \delta\). Choose such a \(\delta\).

Let \(x, z \in \overline{K}\) such that \(d_X(x, z) < \delta/3\), and choose \((x_n) \subset K\) and \((z_n) \subset K\) such that \(x_n \to x\) and \(z_n \to z\). This implies that \(f(x_n) \to \overline{f}(x)\) and \(f(z_n) \to \overline{f}(z)\). There exists \(N_1, N_2, N_3,\) and \(N_4\) such that \(n \geq N_1, n \geq N_2, n \geq N_3,\) and \(n \geq N_4\) implies \(d_X(x, x_n) < \delta/3, d_X(z, z_n) < \delta/3, d_Y(f(x_n), f(x)) < \epsilon/3,\) and \(d_Y(f(z_n), f(z)) < \epsilon/3,\) respectively. Choose \(n \geq \max \{N_1, N_2, N_3, N_4\}\). For such an \(n\), by repeated use of the triangle inequality,

\[
d_X(x_n, z_n) \leq d_X(x_n, x) + d_X(x, z) + d_X(z, z_n) < \delta/3 + \delta/3 + \delta/3 = \delta,
\]

which implies that

\[
d_Y(f(x_n), f(z_n)) < \epsilon/3.
\]

Therefore, by repeated use of the triangle inequality,

\[
d_Y(\overline{f}(x), \overline{f}(z)) \leq d_Y(\overline{f}(x), f(x_n)) + d_Y(f(x_n), f(z_n)) + d_Y(f(z_n), \overline{f}(z)) < \epsilon.
\]

\(\square\)
Part (b)

Two examples are required.

For the first example showing the necessity of $Y$ being complete, suppose $Y = (0, 1)$, $X = \mathbb{R}$, and consider $K = (0, 1) \subset X$ with $f(x) = x$. There is no way to define $f(0)$ and $f(1)$ since any sequence $(x_n) \subset K$ that converges to either 0 or 1 is Cauchy, but not convergent, in $Y$.

For the second example, take $Y = \mathbb{R}$, $X = \mathbb{R}$, $K = (0, 1] \subset X$, and $f(x) = 1/x$, which is easily seen to not have any continuous extension at $x = 0$ since the limit of $f(x)$ as $x$ approaches 0 within $K$ is $+\infty$. 
Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, $X$ compact, and $f : X \to Y$ satisfies two conditions

(i) For each compact set $K \subset X$, $f(K)$ is compact.

(ii) For every nested decreasing sequence of compact sets $(K_n) \subset X$,

$$f(\cap K_n) = \cap f(K_n).$$

Prove that $f$ is continuous.

**Proof.** We prove by contradiction.

Assume that $f$ is not continuous.

Then, there exists an $x \in X$ and $\epsilon > 0$ such that for each $n \in \mathbb{N}$, there exists $x_n \in B_{1/n}(x)$ such that $f(x_n) \notin B_{\epsilon}(f(x))$.

For each $n \in \mathbb{N}$, let $K_n = \overline{B_{1/n}(x)}$ denote the closure of the ball $B_{1/n}(x)$. Since closed subsets of a compact space are compact by a standard result, $K_n$ is compact for each $n \in \mathbb{N}$. By construction, $(K_n)$ is a nested sequence of compact sets and $\cap K_n = \{x\}$.

By assumption (i), $f(K_n)$ is compact for each $n \in \mathbb{N}$ and $(f(K_n))$ is a nested decreasing sequence of compact sets in $Y$ by construction. Since $(f(x_n)) \subset f(K_1)$, there exists a convergent subsequence $(f(x_{n_k}))$. By construction, $(f(x_{n_k}))_{k \geq N} \subset f(K_N)$ for each $N \in \mathbb{N}$, and since $f(K_N)$ are closed (since they are compact) for each $N \in \mathbb{N}$, the limit of $(f(x_{n_k}))$ belongs to $\cap f(K_n)$. By assumption (ii) on $f$, $\cap f(K_n) = f(\cap K_n) = \{f(x)\}$, which implies $f(x_{n_k}) \to f(x)$ contradicting how $(f(x_n))$ was constructed. \qed
(5) Suppose \( f : [-1, 1] \to \mathbb{R} \) is three-times differentiable with continuous third derivative on \([-1, 1]\).

Prove that the series
\[
\sum_{n=1}^{\infty} \left[ n \left( f(1/n) - f(-1/n) \right) - 2f'(0) \right]
\]
converges.

**Proof.** By Taylor’s theorem, for each \( n \in \mathbb{N} \), there exists \( \xi_{n}^{(1)} \in (0, 1/n) \) such that
\[
f(1/n) = f(0) + f'(0) \frac{1}{n} + f''(0) \frac{1}{2n^2} + f'''(\xi_{n}^{(1)}) \frac{1}{6n^3},
\]
and there exists \( \xi_{n}^{(2)} \in (-1/n, 0) \) such that
\[
f(-1/n) = f(0) - f'(0) \frac{1}{n} + f''(0) \frac{1}{2n^2} - f'''(\xi_{n}^{(2)}) \frac{1}{6n^3}.
\]

Then, we have that for each \( n \in \mathbb{N} \), we see that
\[
\left[ n \left( f(1/n) - f(-1/n) \right) - 2f'(0) \right] = \frac{1}{6n^2} \left[ f'''(\xi_{n}^{(1)}) + f'''(\xi_{n}^{(2)}) \right].
\]

Then, since the third derivative is continuous on \([-1, 1]\), it is bounded in magnitude on \([-1, 1]\) by some \( M \geq 0 \), so that
\[
\frac{1}{6n^2} \left| f'''(\xi_{n}^{(1)}) + f'''(\xi_{n}^{(2)}) \right| \leq \frac{M}{3n^2}.
\]

Since
\[
\sum_{n=1}^{\infty} \frac{M}{3n^2}
\]
converges by the integral test, we have that the series converges (in fact converges absolutely). \( \square \)
(6) Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces and $f : X \to Y$. Prove that $f$ is uniformly continuous if and only if for every sequences $(x_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ in $X$ such that $d_X(x_n,z_n) \to 0$ implies $d_Y(f(x_n),f(z_n)) \to 0$.

Proof. First assume that $f$ is uniformly continuous. Let $\epsilon > 0$. There exists $\delta > 0$ such that $d_X(x,z) < \delta$ implies $d_Y(f(x),f(z)) < \epsilon$. Choose such a $\delta > 0$. Consider any sequences $(x_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ in $X$ such that $d_X(x_n,z_n) \to 0$. Then, there exists $N$ such that $n \geq N$ implies $d_X(x_n,z_n) < \delta$, which implies that $d_Y(f(x_n),f(z_n)) < \epsilon$. Thus, $d_Y(f(x_n),f(z_n)) \to 0$.

Now assume that $f$ is not uniformly continuous. Then, there exists $\epsilon > 0$ such that for every $\delta > 0$ there exists $x, z \in X$ with $d_X(x,z) < \delta$ and $d_Y(f(x),f(z)) \geq \epsilon$. Choose such an $\epsilon$, and for each $n \in \mathbb{N}$ let $\delta_n = 1/n$, and choose $x_n, z_n \in X$ such that $d_X(x_n,z_n) < \delta_n$ and $d_Y(f(x_n),f(z_n)) \geq \epsilon$. By construction, $d_X(x_n,z_n) \to 0$ but $d_Y(f(x_n),f(z_n)) \not\to 0$. \qed
(7) Let \( f : [0, 1] \to \mathbb{R} \) be continuously differentiable with \( f(0) = 0 \). Prove that

\[
\left[ \sup \{ |f(x)| : 0 \leq x \leq 1 \} \right]^2 \leq \int_0^1 (f'(x))^2 \, dx.
\]

**Proof.** By the Fundamental Theorem of Calculus (and the fact that \( f(0) = 0 \)), for each \( x \in [0, 1] \),

\[
f(x) = \int_0^x f'(s) \, ds \Rightarrow |f(x)| \leq \int_0^x |f'(s)| \, ds.
\]

By the standard Cauchy-Schwartz (or just Schwartz) inequality

\[
\int_0^x |f'(s)| \, ds \leq \left( \int_0^x |f'(s)|^2 \, ds \right)^{1/2} \left( \int_0^x 1^2 \, ds \right)^{1/2}
\]

\[
\leq \left( \int_0^1 |f'(x)|^2 \, dx \right)^{1/2}.
\]

Thus, for each \( x \in [0, 1] \),

\[
|f(x)| \leq \left( \int_0^1 |f'(x)|^2 \, dx \right)^{1/2}.
\]

Since the inequality holds for all \( x \in [0, 1] \),

\[
\sup \{ |f(x)| : 0 \leq x \leq 1 \} \leq \left( \int_0^1 |f'(x)|^2 \, dx \right)^{1/2}.
\]

Squaring both sides completes the proof. \( \square \)