APPLIED ANALYSIS PRELIMINARY EXAM
JULY 14, 2017

Name: ____

- Exam consists of 7 problems. Do *all* 7 problems. All will be graded.
- Each problem is worth 20 points.
- Justify your solutions: cite theorems that you use, provide counter-examples for disproof, give explanations.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Begin solution to every problem on a new page; write only on one side of a sheet; number all pages throughout; just in case, write your name on every page.
- Do not submit scratch paper.
- Ask the proctor if you have any questions.

Good luck!

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________

Total ____________

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Problems

(1) Let (X, d) be a metric space, $K \subset X$ be nonempty, and let K' denote the set of limit points of K. Define the closure of K as $\overline{K} := K \cup K'$. Prove that $\overline{K}$ is both (1) closed, and (2) if F is closed and $K \subset F$, then $\overline{K} \subset F$. In other words, prove that $\overline{K}$ is the smallest closed set containing K.

(2) Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Prove that if there exists $x \in X$ such that for every subsequence $(x_{n_k})_{k\in\mathbb{N}}$ there exists a subsequence $(x_{n_{k_j}})_{j\in\mathbb{N}}$ such that $x_{n_{k_j}} \to x$, then $x_n \to x$.

(3) Let (X, d_X) and (Y, d_Y) be metric spaces, $K \subset X$ nonempty and open, and $f : K \to Y$. Let $\overline{K}$ denote the closure of K (see problem 1 for definition). Suppose Y is complete and f is uniformly continuous.

  (a) **(15 points)** Prove that there exists a unique uniformly continuous function $\overline{f} : \overline{K} \to Y$ such that $\overline{f}(x) = f(x)$ for every $x \in K$. We call $\overline{f}$ the extension of f to $\overline{K}$.

  (b) **(5 points)** Give (1) an example showing the necessity of the condition that Y is complete, and (2) an example showing that even if Y is complete but f is only continuous, then there may not be an extension of f to $\overline{K}$ that is continuous.

(4) Let (X, d_X) and (Y, d_Y) be metric spaces, X compact, and $f : X \to Y$ satisfies two conditions

  (i) For each compact set $K \subset X$, $f(K)$ is compact.

  (ii) For every nested decreasing sequence of compact sets $(K_n) \subset X$,

$$f(\cap K_n) = \cap f(K_n).$$

Prove that f is continuous.

(5) Suppose $f : [-1, 1] \to \mathbb{R}$ is three-times differentiable with continuous third derivative on $[-1, 1]$. Prove that the series

$$\sum_{n=1}^{\infty} \left[n\big(f(1/n) - f(-1/n)\big) - 2f'(0) \right]$$

converges.

(6) Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. Prove that f is uniformly continuous if and only if for every sequences $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ in X such that $d_X(x_n, z_n) \to 0$ implies $d_Y(f(x_n), f(z_n)) \to 0$.

(7) Let $f : [0, 1] \to \mathbb{R}$ be continuously differentiable with $f(0) = 0$. Prove that

$$\left[\sup\{|f(x)| \,:\, 0 \le x \le 1\}\right]^2 \le \int_0^1 (f'(x))^2\, dx.$$