Solve the following 6 problems.

1. Prove that if series $\sum_{n=1}^{\infty} a_n x^n$ converges for all x such that $|x| < 1$, then the series $\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}$ converges as well if $|x| < 1$.

2. Let X be a nonempty set and d be a metric on X. Prove the standard theorem that the set of all limit points of X is closed.

3. Let X be a nonempty set and d be a metric on X. We say that $K \subset X$ is *sequentially compact* if for every sequence $\{x_n\} \subset K$ there exists a subsequence $\{a_{n_k}\}$ that converges to a point $x \in K$. For a fixed $\epsilon > 0$, we call $\{x_\alpha\}_{\alpha \in \mathcal{A}} \subset X$ an ϵ-net of $K \subset X$ if the family of open balls $\{B_\epsilon(x_\alpha)\}_{\alpha \in \mathcal{A}}$ is an open cover of K. We say that $K \subset X$ is totally bounded if there exists a finite ϵ-net for every $\epsilon > 0$. Use these definitions to prove the standard theorem that a nonempty sequentially compact subset of a metric space is complete and totally bounded.

4. Let X be a nonempty set and d be a metric on X. Suppose f is a continuous function on $A \subset X$ to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Using only the definitions of a set being compact and a function being uniformly continuous, prove that if A is compact, then f is uniformly continuous, and provide a counterexample to the converse.

5. Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$. For a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$, the upper and lower Darboux sums of f on P are defined as

$$U(f, P) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}),$$

and

$$L(f, P) = \sum_{i=1}^{n} \left(\inf_{y \in [x_{i-1}, x_i]} f(y) \right) (x_i - x_{i-1}),$$

respectively. We say that f is Riemann integrable on $[a, b]$ if for every $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

Suppose that $f : [a, b] \to \mathbb{R}$ is bounded. Using the above definitions, prove that if f is Riemann integrable, then f^2 is Riemann integrable, and provide a counterexample to the converse.

Hint: You may find it useful to exploit the fact that for any set A, and any real-valued function f defined on A that $\sup_{x \in A} f(x) - \inf_{y \in A} f(y) = \sup_{x,y \in A} |f(x) - f(y)|$.

6. Let $\mathcal{C}^1([a, b])$ denote the space of real-valued continuously differentiable functions on $[a, b]$ where $a < b$ are real numbers. Define the metric d on $\mathcal{C}^1([a, b])$ as follows (where $f, g \in \mathcal{C}^1([a, b])$)

$$d(f, g) = \sup_{[a,b]} |f(x) - g(x)| + \sup_{[a,b]} |f'(x) - g'(x)|.$$

Suppose $\{f_n\} \subset (\mathcal{C}^1([a, b]), d)$ is a bounded sequence. Prove that if $\{f_n'\}$ is equicontinuous, then there exists a subsequence of $\{f_n\}$ that converges in $(\mathcal{C}^1([a, b]), d)$.