Solve the following 6 problems.

1. Prove that if series $\sum_{n=1}^{\infty} a_n x^n$ converges for all x such that $|x| < 1$, then the series $\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}$ converges as well if $|x| < 1$.

   Solution:

   For $|x| < 1$, $x^n \to 0$ as $n \to \infty$, so there exists an N such that $|x^n| \leq \frac{1}{2}$ for all $n \geq N$. Thus,

   $$\forall n \geq N : \left| a_n \frac{x^n}{1-x^n} \right| \leq \left| a_n \frac{x^n}{1-\frac{1}{2}} \right| = 2\left| a_n x^n \right|$$

   Since $\sum_{n=1}^{\infty} a_n x^n$ converges for all x such that $|x| < 1$, the radius of convergence of this power series is at least 1. Thus, the convergence is absolute for all $|x| < 1$ by a standard result. Furthermore, $2\sum_{n=1}^{\infty} a_n x^n = \sum 2a_n x^n$ also converges absolutely for all $|x| < 1$.

   Therefore, by the comparison test[1], for all $|x| < 1$, the series $\sum_{n=1}^{\infty} a_n \frac{x^n}{1-x^n}$ converges. □

[1] Theorem 3.25(a) in Rudin

2. Let X be a nonempty set and d be a metric on X. Prove the standard theorem that the set of all limit points of X is closed.

Solution 1:

Let A be the set of all limit points of X. We show that A is closed by showing that it contains all of its limit points.

If A is empty, then it is closed by definition. Otherwise, consider any limit point x of A. Since $A \subset X$, this implies that x is a limit point of X, so $x \in A$ by construction. $\square$

Solution 2:

Let A be the set of all limit points of X.

By a standard result, proving A is closed is equivalent to proving that A^c is open. We prove that A^c is open by showing that all of its points are interior to A^c.

If A^c is either equal to the empty set or equal to X, then it is open by definition. Otherwise, consider any $x \in A^c$.

By assumption, x is not a limit point of X, which implies it is an isolated point by definition. This implies there exists an $r > 0$ such that $B_r(x) = \{x\}$, so $B_r(x) \subset A^c$. Thus, x is an interior point of A^c. Since $x \in A^c$ was arbitrary, this proves A^c is open and that A must be closed. $\square$

Solution 3 (proving a more general result):

Let $E \subset X$, E' denote the set of all limit points of E, and E'' denote the set of all limit points from E'. We prove that E' is closed by showing that $E'' \subset E'$.

If E'' is empty, then E' is closed by definition, so assume E'' is not empty and let $x \in E''$.

By definition, for any $r > 0$ there exists $y \in E'$ such that $0 < d(x, y) < r/2$. By definition of E', there exists $z \in E$ such that $0 < d(y, z) < d(x, y) < r/2$ (which implies $z \neq x$). Thus, $0 < d(x, z) \leq d(x, y) + d(y, z) < r$, so x is a limit point of E, which implies $x \in E'$ and it follows that $E'' \subset E'$.

Since $X \subset X$, the result holds for X as well. $\square$

3. Let X be a nonempty set and d be a metric on X. We say that $K \subset X$ is *sequentially compact* if for every sequence $\{x_n\} \subset K$ there exists a subsequence $\{a_{n_k}\}$ that converges to a point $x \in K$. For a fixed $\epsilon > 0$, we call $\{x_\alpha\}_{\alpha \in \mathcal{A}} \subset X$ an ϵ-net of $K \subset X$ if the family of open balls $\{B_\epsilon(x_\alpha)\}_{\alpha \in \mathcal{A}}$ is an open cover of K. We say that $K \subset X$ is totally bounded if there exists a finite ϵ-net for every $\epsilon > 0$. Use these definitions to prove the standard theorem that a nonempty sequentially compact subset of a metric space is complete and totally bounded.

Solution:

Suppose $K \subset X$ is sequentially compact.

Consider any Cauchy sequence $\{x_n\} \subset K$. The sequential compactness of K implies there exists a convergent subsequence $\{x_{n_k}\}$. By a standard result, the Cauchy sequence must also converge to the same limit as the convergent subsequence[2].

Assume K is not totally bounded, so that there exists an $\epsilon > 0$ such that K has no finite ϵ-net.

This means that every finite subset $\{x_1, \ldots, x_n\} \subset K$ has the property that $\cup_{i=1}^n B_\epsilon(x_i)$ is not a cover for K. In other words, $K \backslash (\cup_{i=1}^n B_\epsilon(x_i))$ is not empty.

Pick any $x_1 \in K$, and choose x_2 from $K \backslash B_\epsilon(x_1)$. Having chosen the first n points, choose x_{n+1} from $K \backslash (\cup_{i=1}^n B_\epsilon(x_i))$. This inductively constructs a sequence $\{x_n\} \subset K$ such that $d(x_n, x_m) > \epsilon$ for any $n \neq m$, which implies that any subsequence is *not* Cauchy. Thus, there cannot exist a convergent subsequence, a contradiction of K being sequentially compact. Therefore, K is totally bounded. $\square$

[2]This is also easily proved by an $\epsilon/2$ argument if the result is not remembered.

4. Let X be a nonempty set and d be a metric on X. Suppose f is a continuous function on $A \subset X$ to $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Using only the definitions of a set being compact and a function being uniformly continuous, prove that if A is compact, then f is uniformly continuous, and provide a counterexample to the converse.

Solution:

Suppose A is compact and $f : A \to \mathbb{R}^n$ is continuous.

Let $\epsilon > 0$.

For any $x \in A$, since f is continuous, there exists $\delta_x > 0$ such that for all $y \in B_{\delta_x}(x)$, $||f(x) - f(y)|| < \epsilon/2$.

The set $\{B_{\delta_x/2}(x)\}_{x \in A}$ forms an open cover of A. Since A is compact, there exists a finite subcover that we denote by $\{B_{\delta_{x_i}/2}(x_i)\}_{1 \leq i \leq k}$.

Let $\delta = \min\{\delta_{x_1}/2, \ldots, \delta_{x_k}/2\}$.

Consider any $y, z \in A$ such that $d(y, z) < \delta$. Then, there exists some x_i such that $y, z \in B_{\delta_{x_i}}(x_i)$. By the triangle inequality,

$$||f(y) - f(z)|| \leq ||f(y) - f(x_i)|| + ||f(x_i) - f(z)|| < \epsilon.$$

For a counterexample, suppose that $f : \mathbb{R} \to \mathbb{R}^n$ is the zero map, i.e., $f(x) = \mathbf{0} \in \mathbb{R}^n$ for all $x \in \mathbb{R}$. Then, for any $\epsilon > 0$ and $x, y \in \mathbb{R}$,

$$||f(x) - f(y)|| = 0 < \epsilon,$$

which shows that f is uniformly continuous, but $\mathbb{R}$ is not compact since it is not bounded. □

5. Let $a < b$ be real numbers and $f : [a,b] \to \mathbb{R}$. For a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$, the upper and lower Darboux sums of f on P are defined as

$$U(f,P) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1},x_i]} f(x) \right) (x_i - x_{i-1}),$$

and

$$L(f,P) = \sum_{i=1}^{n} \left(\inf_{y \in [x_{i-1},x_i]} f(y) \right) (x_i - x_{i-1}),$$

respectively. We say that f is Riemann integrable on $[a,b]$ if for every $\epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

Suppose that $f : [a,b] \to \mathbb{R}$ is bounded. Using the above definitions, prove that if f is Riemann integrable, then f^2 is Riemann integrable, and provide a counterexample to the converse.

Hint: You may find it useful to exploit the fact that for any set A, and any real-valued function f defined on A that $\sup_{x \in A} f(x) - \inf_{y \in A} f(y) = \sup_{x,y \in A} |f(x) - f(y)|$.

Solution:

Since f is bounded on $[a,b]$ there exists $M > 0$ such that $|f(x)| < M$ for all $x \in [a,b]$. Thus, for any $x, y \in [a,b]$, we have that

$$|f^2(x) - f^2(y)| = |f(x) - f(y)| \cdot |f(x) + f(y)| \leq 2M|f(x) - f(y)|.$$

Let $\epsilon > 0$.

Since f is Riemann integrable, there exists a partition P on $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon/(2M)$.

Then, for this partition P, we have that

$$\begin{aligned}
U(f^2,P) - L(f^2,P) &= \sum_{i=1}^{n} \left(\sup_{[x_{i-1},x_i]} f^2(x) - \inf_{[x_{i-1},x_i]} f^2(x) \right) (x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} \left(\sup_{x,y \in [x_{i-1},x_i]} |f^2(x) - f^2(y)| \right) (x_i - x_{i-1}) \\
&\leq 2M \sum_{i=1}^{n} \left(\sup_{x,y \in [x_{i-1},x_i]} |f(x) - f(y)| \right) (x_i - x_{i-1}) \\
&= 2M \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1},x_i]} f(x) - \inf_{y \in [x_{i-1},x_i]} f(y) \right) (x_i - x_{i-1}) \\
&= 2M[U(f,P) - L(f,P)] \\
&< \epsilon.
\end{aligned}$$

For a counterexample, suppose $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = 1$ whenever $x \in [0,1] \cap \mathbb{Q}$, and $f(x) = -1$ otherwise. Then, $f^2(x) = 1$ for all $x \in [0,1]$, which is clearly Riemann integrable, but, for any partition P of $[0,1]$, we have $U(f,P) = 1 \neq -1 = L(f,P)$, so f is not Riemann integrable. $\square$

6. Let $\mathcal{C}^1([a,b])$ denote the space of real-valued continuously differentiable functions on $[a,b]$ where $a < b$ are real numbers. Define the metric d on $\mathcal{C}^1([a,b])$ as follows (where $f, g \in \mathcal{C}^1([a,b])$)

$$d(f,g) = \sup_{[a,b]} |f(x) - g(x)| + \sup_{[a,b]} |f'(x) - g'(x)|.$$

Suppose $\{f_n\} \subset (\mathcal{C}^1([a,b]), d)$ is a bounded sequence. Prove that if $\{f_n'\}$ is equicontinuous, then there exists a subsequence of $\{f_n\}$ that converges in $\mathcal{C}^1([a,b]), d)$.

Solution:

Since $\{f_n\}$ is bounded in $\mathcal{C}^1([a,b])$, there exists an M such that $\{f_n\} \subset B_M(0)$ (where 0 indicates the zero function). By the definition of the metric, this implies that both $\{f_n\}$ and $\{f_n'\}$ are uniformly bounded.

Since $\{f_n'\}$ is equicontinuous, the Arzelà-Ascoli theorem implies that there exists a subsequence $\{f_{n_k}'\}$ that converges uniformly to $g \in \mathcal{C}([a,b])$.

Since $\{f_{n_k}\}$ is uniformly bounded, $\{f_{n_k}(a)\}$ is a bounded set of real numbers, so there exists a convergent subsequence $\{f_{n_{k_l}}(a)\}$. Let $f(a)$ define this limit.

By the fundamental theorem of calculus, for each l,

$$f_{n_{k_l}}(x) = f_{n_{k_l}}(a) + \int_a^x f_{n_{k_l}}'(y)\, dy.$$

Since $\{f_{n_{k_l}}'\}$ also converges uniformly to g, then we can take the limit as $l \to \infty$ above for each $x \in [a,b]$ and bring the limit through the integral by a standard result to define $f(x)$ by

$$f(x) = \lim_{l \to \infty} f_{n_{k_l}}(x) = f(a) + \int_a^x g(y)\, dy.$$

By the fundamental theorem of calculus, we have that $f' = g(x)$, so $f \in \mathcal{C}^1([a,b])$. $\square$