University of Colorado Denver
Department of Mathematical and Statistical Sciences
Applied Analysis Ph.D. Preliminary Exam
July 11, 2010

Name: ____

Exam Rules:

- This is a closed book exam. Once the exam begins, you have 4 hours to do your best. All solutions will be graded and your final grade will be based on the total of all of them.
- Each problem is worth 20 points; parts of problems have equal value unless said otherwise.
- Justify your solutions: **cite theorems that you use**, provide counter-examples for disproof, give explanations, and show calculations for numerical problems.
- If you are asked to prove a theorem, do not merely quote that theorem as your proof; instead, produce an independent proof.
- Begin each solution on a new page and use additional paper, if necessary.
- Write only on one side of paper.
- Write legibly using a dark pencil or pen.
- Ask the proctor if you have any questions.

Good luck!

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DO NOT TURN THE PAGE UNTIL TOLD TO DO SO.

1. A function $f : \mathbb{R} \to \mathbb{R}$ is called additive if $f(x+y) = f(x) + f(y)$ for all real x, y. Show that if f is additive and continuous, then $f(x) = cx$ for some real constant c.

   Solution: Let f be additive. First we show that

   $$f(na) = nf(a) \text{ for any integer } n \text{ and any real } a.$$

   The statement holds trivially for $n = 1$. Suppose it holds for some integer $n > 0$. Then

   $$f((n+1)a) = f(na + a) = f(na) + f(a) = nf(a) + f(a) = (n+1) f(a),$$

   which completes the proof by induction for $n \in \mathbb{N}$. Further, $f(0) = 0$ because

   $$f(0) = f(0+0) = f(0) + f(0).$$

   For integer $n < 0$, we have

   $$0 = f(0) = f(an - an) = f(na) + f((-n)a) = f(na) + (-n) f(a),$$

   so again $f(na) = nf(a)$. Next, for any integer $q \neq 0$, setting $a = 1/q$, we have that

   $$qf\left(\frac{1}{q}\right) = f\left(\frac{q}{q}\right) = f(1),$$

   so

   $$f\left(\frac{1}{q}\right) = \frac{f(1)}{f(q)},$$

   and finally for any integers p and $q \neq 0$.

   $$f(\frac{p}{q}) = pf\left(\frac{1}{q}\right) = f(1)\frac{p}{q}.$$

   So, set $c = f(1)$, and let $r \in \mathbb{R}$. We will show that $f(r) = cr$ by continuity. By the density of rationals, there exists a sequence of rational numbers $r_n \to r$. Since f is continuous,

   $$f(r) = \lim_{n\to\infty} f(r_n) = \lim_{n\to\infty} f(1) r_n = f(1) \lim_{n\to\infty} r_n = cr.$$

2. Find an example of a metric space (V, d) and a set $A \subset V$ such that A is closed and bounded but not compact.

   Make sure you actually formulate the definitions and prove that your set A is closed, bounded, and not compact in your metric space. Simply an example without detailed proofs is insufficient.

   Solution: Let us consider a well-known metric space, (V, d), called the "discrete metric space" where $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise. All axioms of a metric space:

   (a) $d(x, y) \geq 0$ (non-negativity)

   (b) $d(x, y) = 0$ if and only if $x = y$ (identity of indiscernibles)

   (c) $d(x, y) = d(y, x)$ (symmetry)

   (d) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

   are trivially satisfied. Let us take $A = V$.

   A set is called closed if its complement is open. An empty set is open by definition. Thus A is closed, as every metric space, as a set, is closed in itself, since its complement is empty.

   A set $A \subset V$ such that A is called bounded if there exist a positive constant C and x—an element of V—such that $\sup_{y \in A} d(x, y) < C$. Taking $C = 2$ we observe that every set in (V, d) is bounded, including $A = V$.

   There are several equivalent definition of compactness in metric spaces. For our purpose it is convenient to consider the so-called "sequential compactness:" a set A is called compact (sequentially compact) if from any sequence of elements of A one can choose a convergent in (V, d) subsequence.

   Let us now specify the number of elements in V by setting it to be an infinite countable set. Let us consider a sequence made of all elements of $A = V$. Since the distance between any two elements with distinct distinct indexes is one, there is no convergent in (V, d) subsequence. Which means that A is not compact.

3. If $\{a_n\}$ is a convergent sequence of real numbers, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Prove, or find a counterexample.

Solution: Let

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Since $\{a_n\}$ converges (to A), then $\{a_n - A\}$ converges to 0. This implies that there exists a number, positive, c such that $|a_n - A| < c$ for all n. Let $\epsilon > 0$ such that $c > \epsilon$. Define $\delta = \frac{\epsilon}{2}$. Then there exists a natural number M such that $n > M$ implies that $|a_n - A| < \delta$. Let $M_1 = \frac{M(c-\delta)}{\delta}$. This implies that for $n > \max\{M, M_1\}$,

$$\begin{aligned}
|b_n - A| &= |\frac{1}{n}\sum_{k=1}^{n} a_k - A| \\
&= \frac{1}{n}|\sum_{k=1}^{n}(a_k - A)| \\
&\leq \frac{1}{n}\sum_{k=1}^{n}|a_k - A| \\
&= \frac{1}{n}\left[\sum_{k=1}^{M}|a_k - A| \quad + \sum_{k=M+1}^{n}|a_k - A|\right].
\end{aligned}$$

But we know that $|a_k - A| < c$ for all k.and $|a_k - A| < \delta$ for $k > M$, so

$$\begin{aligned}
|b_n - A| &< \frac{1}{n}\left[Mc + (n - M)\delta\right] \\
&= \frac{M(c-\delta)}{n} + \delta \\
&< \delta + \delta \\
&= \epsilon.
\end{aligned}$$

Thus $b_n \to A$.

4. For a real variable $x \in [-1, 1]$ let $D(x)$ be a function that takes the value 1 if x is rational and the value 0 otherwise. Is the function $F(x) = xD(x)$ Riemann integrable? If so, what is the value of the integral $\int_{-1}^{1} F(x)dx$?

Solution. The function $F(x) = xD(x)$ in not Riemann integrable. The proof is essentially the same as the proof that function $D(x)$ in not Riemann integrable. We consider an arbitrary partitioning of the interval $[-1, 1]$ of integration into subintervals. Let us consider one of these subintervals, $[x_i, x_{i+1}]$.

If $x_i \geq 0$ the contribution from this interval to the upper Riemann sum is $x_{i+1}(x_{i+1} - x_i)$, since the $\sup_{x \in [x_i, x_{i+1}]} F(x) = x_{i+1}$. Indeed, first, we observe that $F(x) \leq x_{i+1}$ for $x \in [x_i, x_{i+1}]$. Second, if x_{i+1} is rational, than simply $F(x_{i+1}) = x_{i+1}$. If x_{i+1} is irrational, than there exists a sequence $y^{(j)}$, $j = 1, \ldots, \infty$ of rational numbers such that $y^{(j)} \in [x_i, x_{i+1}]$ and $y^{(j)} \to x_{i+1}$ as $j \to \infty$, because of the density of rationals on the real line. E.g., one can use a decimal representation of x_{i+1} chopped at the j-th decimal digit to construct a specific example of the sequence $y^{(j)}$, $j = 1, \ldots, \infty$. Since every $y^{(j)}$ is rational, we have $F(y^{(j)}) = y^{(j)}$, but $y^{(j)} \to x_{i+1}$, so $F(y^{(j)}) \to x_{i+1}$.

At the same time, if still $x_i \geq 0$, the contribution from this interval to the lower Riemann sum is zero. Indeed, first, we observe that $F(x) \geq 0$ for $x \in [x_i, x_{i+1}]$. But any nonempty interval on a real line contains at least one irrational point, by the density of irrationals on the real line. Thus, $\min_{x \in [x_i, x_{i+1}]} F(x) = 0$.

Similarly, if now $x_{i+1} \leq 0$, the contribution from this interval to the upper Riemann sum is zero, while the contribution from this interval to the lower Riemann sum is $x_i(x_{i+1} - x_i)$.

If there is an interval such that $x_i < 0 < x_{i+1}$, its contribution to either sum in absolute value is bounded by $\max\{-x_i, x_{i+1}\}(x_{i+1} - x_i)$, where $x_{i+1} - x_i \to 0$ as we refine the partition. So it can be ignored.

The Riemann sums on $[-1, 1]$ can be found by computing separately the Riemann sums on $[-1, 0]$ and $[0, 1]$ and adding them, by additivity of Riemann sums.

We first deal with the upper Riemann sum. Let us notice that the upper Riemann sum of $F(x) = xD(x)$ for $x \in [0, 1]$ is exactly the same as the upper Riemann sum of the Riemann-integrable function $f(x) = x$. Thus, summing up, and taking the limit of the upper Riemann sum of $F(x)$ for $x \in [0, 1]$ gives us the same number as simply $\int_0^1 f(x)dx = \int_0^1 xdx = 1/2$. The other term, of the upper Riemann sum of $F(x)$ for $x \in [-1, 0]$ is zero, therefore the upper Riemann sum on $[-1, 1]$ for $F(x)$ is equal to $1/2 + 0 = 1/2$.

Similar arguments show that the lower Riemann sum is $\int_{-1}^{0} xdx = -1/2$. Since the upper and lower Riemann sums are different, the function $F(x)$ in not Riemann integrable.

5. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$, let

$$f_n(x) = \frac{x}{1+nx^2}.$$

(a) Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a function f.

(b) Show that the sequence of derivatives $\{f_n'\}$ does not converge uniformly on $\mathbb{R}$ to any function.

Solution.

(a) Let $\varepsilon > 0$. Clearly,

$$f(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0.$$

To prove that $f_n \rightrightarrows 0$ on $\mathbb{R}$, we need to show that there is M such that $|f_n(x)| < \varepsilon$ for all $x \in \mathbb{R}$ and all $n > M$. First,.

$$\left|\frac{x}{1+nx^2}\right| \le |x| < \varepsilon \quad \text{if } |x| < \varepsilon.$$

and

$$\left|\frac{x}{1+nx^2}\right| \le \left|\frac{x}{nx^2}\right| \le \frac{1}{n\varepsilon} \text{ if } |x| \ge \varepsilon.$$

Thus,

$$\left|\frac{x}{1+nx^2}\right| < \varepsilon \quad \forall n > \frac{1}{\varepsilon^2}$$

and we can take $M = \lceil 1/\varepsilon^2 \rceil$.

(b) The theorem about differentiation of functional sequences states that if $\{f_n'\}$ exists and converges uniformly on an interval $[a,b]$ and $f_n(x_0)$ converges at some $x_0 \in [a,b]$, then there exists a differentiable function ϕ such that $f_n \to \phi$ uniformly on $[a,b]$ and $f_n' \to \phi'$ uniformly on $[a,b]$. Let us take $[a,b] = [-1,1]$. We already know from part (a) that $f_n(x_0)$ converges (to zero), e.g., at $x_0 = 0$.

We prove by contradiction. Let us assume that $\{f_n'\}$ converges uniformly on $\mathbb{R}$, thus it also converges uniformly on $[a,b] = [-1,1]$. (It would be an error to use such a contradiction on $\mathbb{R}$ directly since the theorem is formulated for a bounded closed interval only.)

The assumptions of the theorem above are satisfied, so there exists a differentiable function ϕ such that $f_n \to \phi$ uniformly on $[a,b]$ and $f_n' \to \phi'$ uniformly on $[a,b]$. On the one hand, from part (a), $f_n \to f \equiv 0$ uniformly on on $\mathbb{R}$ and thus on $[a,b]$, so $\phi = f \equiv 0$ by the uniqueness of the limit, and clearly $\phi' \equiv 0$. On the other hand, by direct calculation,

$$\begin{aligned} f_n'(x) &= \left(\frac{x}{1+nx^2}\right)' = \frac{\left(1+nx^2\right)x' - \left(1+nx^2\right)' x}{\left(1+nx^2\right)^2} \\ &= \frac{1+nx^2-2nx^2}{\left(1+nx^2\right)^2} = \frac{1-nx^2}{\left(1+nx^2\right)^2}. \end{aligned}$$

Thus,

$$\lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty} 1 = 1 \neq 0,$$

which is a contradiction.