Applied Analysis Preliminary Examination—January 2016

Name:

- Turn in problems 1 to 4, and exactly two out of problems 5,6,7. Only 6 solutions will be graded. Each problem is worth 20 points.
- Be sure to show all your relevant work. Rewrite your solutions, if necessary, so they are neat and easy to read.
- **Only write on one side of each sheet.**
- Start a new sheet of paper for every problem, copy the entire problem statement, and write your name and the problem number on every sheet. Number the pages within each problem.
- Justify your solutions.
- If you use a theorem from Rudin or class, state it. If you are unsure if a statement must be proved or may merely be stated, ask your friendly proctor.

1	2	3	4	5	6	7	$\sum$

Section 1: Complete ALL four of the following questions.

1. Let (a_n) and (b_n) be bounded nonnegative sequences. Prove that

$$\limsup_{n\to\infty} a_n b_n \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right).$$

Solution 1.. We use the definition that lim sup of a sequence is the supremum of all subsequence limits. Since the sequences here are bounded, we do not need to consider subsequences with infinite limits, so let $(a_{n_k} b_{n_k})$ be a convergent subsequence of $(a_n b_n)$. Since (a_n) is bounded, there exists a convergent subsequence $\left(a_{n_{k_l}}\right)$. Since $\left(b_{n_{k_l}}\right)$ is bounded, we can select from it further a convergent subsequence $\left(b_{n_{k_{l_m}}}\right)$. Since the limit of a subsequence is the same as the limit of the sequence it was selected from, we have

$$\lim_{k\to\infty} a_{n_k} b_{n_k} = \lim_{m\to\infty} a_{n_{k_{l_m}}} b_{n_{k_{l_m}}} = \lim_{m\to\infty} a_{n_{k_{l_m}}} \lim_{m\to\infty} b_{n_{k_{l_m}}} \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right).$$

Because $(a_{n_k} b_{n_k})$ was an arbitrary convergent subsequence of $(a_n b_n)$, it follows that

$$\limsup_{n\to\infty} a_n b_n = \sup\left\{\lim_{k\to\infty} a_{n_k} b_{n_k} \,|\, (a_{n_k} b_{n_k}) \text{ converges}\right\} \leq \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right).$$

Solution 2. We use the equivalent definition

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} s_n, \quad s_n = \sup\{x_n, x_{n+1}, \ldots\}.$$

Define the sets

$$A_n = \{a_n, a_{n+1}, \ldots\}, \quad B_n = \{b_n, b_{n+1}, \ldots\}, \quad C_n = \{a_n b_n, a_{n+1} b_{n+1}, \ldots\}.$$

By the definition of lim sup, we have

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} (\sup A_n), \quad \limsup_{n\to\infty} b_n = \lim_{n\to\infty} (\sup B_n), \quad \limsup_{n\to\infty} a_n b_n = \lim_{n\to\infty} (\sup C_n). \tag{1}$$

Fix n. We will estimate $\sup C_n$ terms of $\sup A_n$ and $\sup B_n$. Because $\sup A_n$ is an upper bound on A_n, we have

$$\forall k \geq n : 0 \leq a_k \leq \sup A_n$$

and similarly

$$\forall k \geq n : 0 \leq b_k \leq \sup B_n$$

Consequently,

$$\forall k \geq n : 0 \leq a_k b_k \leq (\sup A_n)(\sup B_n).$$

Thus, $(\sup A_n)(\sup B_n)$ is an upper bound on C_n. Because $\sup C_n$ is the least upper bound on C_n, we conclude that

$$\sup C_n \leq (\sup A_n)(\sup B_n).$$

Now taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \sup C_n \le \lim_{n\to\infty} (\sup A_n)(\sup B_n) = \left(\lim_{n\to\infty} \sup A_n\right)\left(\lim_{n\to\infty} \sup B_n\right),$$

where all limits exists because they are limits of monotone sequences. Thus, using (1), we conclude that

$$\limsup_{n\to\infty} a_n b_n \le \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right)$$

as desired. Note: Justifications required for full credit, such as using that $a_k, b_k \ge 0$, that limits exists and why, supremum is an upper bound, and the least upper bound, in appropriate places.

Solution 3. We use the equivalent definition that for a bounded sequence x_n, $\limsup_{n\to\infty} x_n$ is the smallest number x with the property

$$\forall \varepsilon > 0 \exists N \forall n > N : x_n < x + \varepsilon. \tag{2}$$

The sequences (a_n), (b_n) are bounded, so $a = \limsup_{n\to\infty} a_n$ and $b = \limsup_{n\to\infty} b_n$ are real (that is, finite). Let $\varepsilon > 0$. Then there exists $\varepsilon' > 0$ such that $(a+\varepsilon')(b+\varepsilon') < ab + \varepsilon$ (from the continuity of multiplication, $\lim_{\varepsilon'\to 0}(a+\varepsilon')(b+\varepsilon') = ab$). From (2),

$$\exists N_a \forall n > N_a : a_n < a + \varepsilon'$$
$$\exists N_b \forall n > N_b : b_n < b + \varepsilon'$$

Then for all $n > N = \max\{N_a, N_b\}$, we have

$$a_n b_n < \left(a+\varepsilon'\right)\left(b+\varepsilon'\right) < ab + \varepsilon$$

(using $a_n \ge 0$, $b_n \ge 0$). Since $\varepsilon > 0$ was arbitrary, $\limsup_{n\to\infty} a_n b_n \le ab$.

2. Suppose that (M, d) is a compact metric space. Prove that for every $\varepsilon > 0$, there exists a finite set $A \subset M$ such that the distance of every point in M to A is less than ε.

   Solution. The balls $B_\varepsilon(x) = \{y \in M | d(x,y) < \varepsilon\}$, $x \in M$, are open because every ball in a metric space is open and they cover M, because $d(x,x) = 0$, so $x \in B_\varepsilon(x)$, hence $\{x\} \subset B_\varepsilon(x)$, and

   $$M = \bigcup_{x\in M} \{x\} \subset \bigcup_{x\in M} B_\varepsilon(x).$$

   That is, $\{B_\varepsilon(x)\}_{x\in M}$ is an open cover of M. By the definition of compact metric space, there exists a finite subcover $B_\varepsilon(x_i)$, $i = 1, \ldots, n$,

   $$M \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i).$$

   Every point $x \in M$ is in at least one of the balls $B_\varepsilon(x_i)$, then $d(x, x_i) < \varepsilon$. Thus, $A = \{x_1, \ldots, x_n\}$ is the desired set.

3. Let f and g be continuous maps of a metric space (X, d_X) into a metric space (Y, d_Y). Let $h : X \to \mathbb{R}$ be defined by $h(x) = d_Y(f(x), g(x))$. Prove h is continuous and that the set $\{x \in X : f(x) = g(x)\}$ is closed.

   Solution. First note that by the generalized triangle inequality (or just repeating the triangle inequality) that for any $x, z \in X$,

$$\begin{aligned} d_Y(f(x), g(x)) &\leq d_Y(f(x), f(z)) + d_Y(f(z), g(z)) + d_Y(g(z), g(x)) \\ \Rightarrow d_Y(f(x), g(x)) - d_Y(f(z), g(z)) &\leq d_Y(f(x), f(z)) + d_Y(g(x), g(z)). \end{aligned}$$

   By changing the roles of x and z, we see that

$$|d_Y(f(x), g(x)) - d_Y(f(z), g(z))| \leq d_Y(f(x), f(z)) + d_Y(g(x), g(z)).$$

   Consider any $x \in X$. Let $\epsilon > 0$. Since f and g are continuous on X, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $z \in X$ with $d_X(x, z) < \delta_1$, $d_Y(f(x), f(z)) < \epsilon/2$ and for all $z \in X$ with $d_X(x, z) < \delta_2$, $d_Y(g(x), g(z)) < \epsilon/2$. Let $\delta = \min\{\delta_1, \delta_2\}$.

   For any $z \in X$ with $d_X(x, z) < \delta$, we then have

$$|h(x) - h(z)| = |d_Y(f(x), g(x)) - d_Y(f(z), g(z))| \leq d_Y(f(x), f(z)) + d_Y(g(x), g(z)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

   Thus, h is continuous.

   Since h is continuous, a standard theorem states that the inverse image of a closed set is closed. Thus, $h^{-1}(\{0\}) = \{x \in X : h(x) = 0\} = \{x \in X : f(x) = g(x)\}$ is a closed set.

4. Let $(\mathcal{C}([a, b]), d)$ denote the metric space of continuous functions on $[a, b]$, where $a < b$ are real numbers, and $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$. Let $(f_n) \subset \mathcal{C}([a, b])$ be a uniformly equicontinuous sequence of functions that converge pointwise to f on $[a, b]$. Prove that f is continuous on $[a, b]$.

   Solution. Let $\epsilon > 0$. Since the sequence (f_n) is uniformly equicontinuous, there exists $\delta > 0$ such that for all $x, y \in [a, b]$ with $|x - y| < \delta$, we have $|f_n(x) - f_n(y)| < \epsilon/3$ for all n. Let $x, y \in [a, b]$ with $|x - y| < \delta$. Since $f_n(x) \to f(x)$ and $f_n(y) \to f(y)$, there exists N_1 and N_2 such that for all $n > N_1$, $|f_n(x) - f(x)| < \epsilon/3$ and for all $n > N_2$, $|f_n(y) - f(y)| < \epsilon/3$. Choose $n = \max\{N_1, N_2\} + 1$, then by repeated application of the triangle inequality,

$$|f(x) - f(y)| < |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \epsilon.$$

   Another solution. Since (f_n) converge pointwise, they are pointwise bounded. Since the interval $[a, b]$ is compact, and (f_n) are pointwise bounded and uniformly equicontinuous, there exists a uniformly convergent subsequence (f_{n_k}). Since uniform convergence implies pointwise convergence, (f_{n_k}) and (f_n) converge pointwise to the same limit, thus f_{n_k}converges uniformly to f. Since the the limit of uniformly convergence sequence of continuous functions is continuous, f is continuous.

Section 2: Complete exactly TWO of the following three questions. If you submit three problems, only the first two will be graded.

5. Prove that there exists exactly one $x \in [1, +\infty)$ such that $x = 1 + \sin \frac{x}{2}$, using the contraction theorem (also known as the Banach contraction principle). Verify all assumptions of the theorem.

Solution. The problem is of the form $x = f(x)$ with $f(x) = 1 + \sin \frac{x}{2}$. To apply the contraction theorem, we need to verify that with a suitable choice of S, (i) $f : S \to S$ (ii) f is a contraction (iii) S is complete. We cannot choose $S = [1, \infty)$ because it is not true that $f : [1, \infty) \to [1, \infty)$; for example, $3\pi \in [1, \infty)$ but $f(\pi) = 1 + \sin \frac{3\pi}{2} = 1 - 1 = 0 \notin [1, \infty)$. So choose $S = \mathbb{R}$, then (i) is satisfied. To show (ii) that f is a contraction, because f is differentiable, for any $x, y \in \mathbb{R}$

$$f(x) - f(y) = f'(\xi)(x - y)$$

for some ξ between x and y by the mean value theorem. Now $f'(x) = \frac{1}{2} \cos \frac{x}{2}$, thus $|f'(\xi)| \leq \frac{1}{2}$, which gives

$$|f(x) - f(y)| \leq \frac{1}{2} |x - y| .$$

To show (iii) just note that $\mathbb{R}$ is complete. So, from the Banach contraction principle, there is a unique $x^* \in \mathbb{R}$ such that $x^* = 1 + \sin \frac{x^*}{2}$. It remains to show that $x \in [1, +\infty)$. (Draw a picture, then it is clear, but we need to actually prove this). Consider the function $g(x) = x - f(x)$. We have

$$g(1) = 1 - f(1) = 1 - \left(1 - \sin \frac{1}{2}\right) = -\sin \frac{1}{2} < 0,$$

because $\frac{1}{2} \in (0, \pi)$, and

$$g(3) = 3 - \left(1 + \sin \frac{3}{2}\right) = 2 - \sin \frac{3}{2} > 0,$$

because $|\sin x| \leq 1$. Since $g(x) = x - f(x)$ is continuous, by the intermediate value theorem, there exists a solution of $x - f(x) = 0$ in $(1, 3)$; because the solution of $x - f(x) = 0$ is unique, $x^* \in (1, 3) \subset [1, \infty)$.

6. Let (g_k) be a sequence of real-valued functions defined on $S \subset \mathbb{R}$. If $\sum_{k=1}^{\infty} g_k$ converges uniformly on S to real-valued function g, prove that $g_k \to 0$ uniformly on S as $k \to \infty$.

Solution. Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} g_k$ converges uniformly on S, then it satisfies the Cauchy criterion uniformly. Thus, there exists K such that for all $n, m > K$, with $n \geq m$,

$$\left| \sum_{k=m}^{n} g_k(x) \right| < \epsilon \ \forall x \in S.$$

Choosing $n = m > K$ above shows that for all $n > K$,

$$|g_n(x)| < \epsilon \ \forall x \in S.$$

Another solution. Uniform convergence of $\sum_{k=1}^{\infty} g_k = g$ means that the partial sums $s_n = \sum_{k=1}^{n} g_k \to g$ uniformly. Then $g_n = s_n - s_{n-1} \to g - g = 0$ uniformly.

7. Prove the following theorem.

Suppose (f_n) is a sequence of real-valued continuous functions on the interval $[a, b]$, $a < b$, and the derivatives f_n' are continuous on $[a, b]$. If

(a) the sequence of derivatives (f_n') converges uniformly on $[a, b]$ to $g : [a, b] \to \mathbb{R}$, and

(b) there exists a point $x_0 \in [a, b]$ such that $\lim_{n\to\infty} f_n(x_0)$ exists,

then the functions f_n converge uniformly to a differentiable function f on $[a, b]$ such that $f' = g$ on $[a, b]$.

Solution. Since the functions f_n are continuously differentiable on $[a, b]$, the Fundamental Theorem of Calculus implies that for any $x \in [a, b]$, we have

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(s)\, ds.$$

Define $f(x)$ for any $x \in [a, b]$ by

$$f(x) = f(x_0) + \int_{x_0}^{x} g(s)\, ds.$$

Since $f_n' \to g$ uniformly on $[a, b]$ and the f_n' are continuous on $[a, b]$ for all n, and uniform limit of a sequence of continuous functions is continuous, it follows that g is continuous on $[a, b]$. Then, by the Fundamental Theorem of Calculus, f is continuously differentiable on $[a, b]$ and $f' = g$ on $[a, b]$.

It remains to show that $f_n \to f$ uniformly on $[a, b]$. Let $x \in [a, b]$. Since $f_n' \to g$ uniformly, we have from the linearity of the integral and standard inequalities,

$$\begin{aligned} |f_n(x) - f(x)| &= \left| f_n(x_0) + \int_{x_0}^{x} f_n'(s)\, ds - f(x_0) - \int_{x_0}^{x} g(s)\, ds \right| \\ &\le |f_n(x_0) - f(x_0)| + \int_{x_0}^{x} \left| f_n'(s) - g(s) \right| \, ds \\ &\le |f_n(x_0) - f(x_0)| + (b-a) \sup_{s\in[a,b]} \left| f_n'(s) - g(s) \right| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since the right hand side is independent of x, the conclusion follows.