January 2018 Preliminary exam in Applied Analysis

February 7, 2018

Solve all of Problems 1 to 4, and two out of the problems 5, 6, 7.

1. Let (X, d_X) and (Y, d_Y) be metric spaces and $K \subset X$ be compact. Prove the standard result that if $f : X \to Y$ is continuous, then $f(K) \subset Y$ is compact.

   Solution. Let $\{V_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$, that is, all V_α are open in Y and $f(K) \subset \bigcup_{\alpha \in A} V_\alpha$. From $f(K) \subset \bigcup_{\alpha \in A} V_\alpha$, it follows that

   $$K \subset f^{-1}(f(K)) \subset f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha)$$

   Since f is continuous, all $f^{-1}(V_\alpha)$ are open. Since K is compact, there exists finite $I \subset A$ such that

   $$K \subset \bigcup_{\alpha \in I} f^{-1}(V_\alpha).$$

   Then

   $$f(K) \subset \bigcup_{\alpha \in I} f\left(f^{-1}(V_\alpha)\right) = \bigcup_{\alpha \in I} V_\alpha.$$

   Thus, every open cover of $f(K)$ has a finite subcover, so $f(K)$ is compact.

   (Note that while $f\left(f^{-1}(V_\alpha)\right) = V_\alpha$, in general only $K \subset f^{-1}(f(K))$ - what if f is not one-to-one.)

2. Let (X, d) be a metric space, $A \subset X$ nonempty, and for any $x \in X$, define the distance from x to A as

   $$\operatorname{dist}(x, A) = \inf\{d(x, a) : a \in A\}.$$

   We say that y is a limit point of A if for all $r > 0$ there exists $a \in A$ such that $d(a, y) < r$. Using this definition of a limit point, show that y is a limit point of A if and only if $\operatorname{dist}(y, A) = 0$.

   Solution.

$\Rightarrow$: Assume that y is a limit point of A. For every $r > 0$, there exists $a \in A$ such that $d(a, y) < r$, thus $\text{dist}(x, A) \leq d(a, y) < r$. Consequently, $\text{dist}(x, A) \leq 0$, and since $\text{dist}(x, A) \geq 0$ because metric is nonnegative, we have $\text{dist}(x, A) = 0$.

$\Leftarrow$:Assume that $\text{dist}(x, A) = 0$. Let $r > 0$. Since $\inf\{d(x, a) : a \in A\} = 0$, from properties of infimum, there exists $a \in A$ such that $d(x, a) < r$. Otherwise r would be lower bound on $\{d(x, a) : a \in A\}$, and since infimum is largest lower bound, $\inf\{d(x, a) : a \in A\} \geq r > 0$).

3. Let $a < b$ be real numbers and (f_n) be a sequence of contraction maps such that $f_n : [a, b] \to [a, b]$ for all n, prove the following:

   (a) There exists a uniformly convergent subsequence (f_{n_k}).

   (b) If f denotes the limit of the uniformly convergent subsequence, then there exists $x \in [a, b]$ such that $f(x) = x$.

Solution.

(a) By assumption, f_n are contraction maps, that is, there exist $c_n < 1$ such that

$$\forall n \forall x, y \in [a, b] : |f_n(x) - f_n(y)| \leq c_n |x - y| .$$

In particular,.

$$\forall n \forall x, y \in [a, b] : |f_n(x) - f_n(y)| \leq |x - y| .$$

Therefore, the functions $\{f_n\}$ are uniformly equicontinuous: choose $\delta = \varepsilon$ in

$$\forall \varepsilon > 0 \exists \delta > 0 \forall n \forall x, y \in [a, b] : |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon.$$

Since $f_n(x) \in [a, b]$, the functions $\{f_n\}$ are also uniformly bounded. Since $[a, b]$ is compact, by the Arzela-Ascoli theorem, there exists a uniformly convergent subsequence (f_{n_k}).

(b) Fix k. Since each f_{n_k} is a contraction, $f_{n_k} : [a, b] \to [a, b]$, and the closed interval $[a, b]$ is a closed subset of $\mathbb{R}$, which is complete, so $[a, b]$ is complete, the Banach contraction theorem applies, and there exists $x_{n_k} \in [a, b]$ such that $f_{n_k}(x_{n_k}) = x_{n_k}$. Since $[a, b]$ is compact, there exists a convergent subsequence (x_{n_ℓ}), $\{n_\ell\} \subset \{n_k\}$. Denote $x = \lim_{\ell \to \infty} x_{n_\ell}$. We will show that $f(x) = x$. By triangle inequality,

$$\begin{aligned} |f(x) - x| &\leq |f(x) - f_{n_\ell}(x)| + |f_{n_\ell}(x) - f_{n_\ell}(x_{n_\ell})| + |f_{n_\ell}(x_{n_\ell}) - x_{n_\ell}| + |x_{n_\ell} - x| \\ &\leq |f(x) - f_{n_\ell}(x)| + |x_{n_\ell} - x| + 0 + |x_{n_\ell} - x| \\ &= |f(x) - f_{n_\ell}(x)| + 2|x_{n_\ell} - x| . \end{aligned}$$

Let $\varepsilon > 0$. Since $f_{n_\ell}(x) \to f(x)$ and $x_{n_\ell} \to x$, there exists N_1 such that for all $n \geq N_1$, $|f(x) - f_{n_\ell}(x)| < \varepsilon/3$ and N_2 such that $n \geq N_2$, $|x - x_{n_\ell}| < \varepsilon/3$. Then $|f(x) - x| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $|f(x) - x| = 0$.

4. Let (X, d) be a compact metric space and (f_n) a sequence of continuous real-valued functions defined on X that converge pointwise to a continuous function f. Prove the standard result: if $f_n(x) \geq f_{n+1}(x)$ for all $x \in X$ and $n \in \mathbb{N}$, then (f_n) converge uniformly.

   Solution. Define $g_n(x) = f_n(x) - f(x) \geq 0$. Let $\varepsilon > 0$. Then, for each $x \in X$, $\lim_{n\to\infty} g_n(x) = 0$, so there exists N_x such that $g_{N_x}(x) < \varepsilon/2$. Since g_{n_x} is continuous, there exists δ_x such that if $d(x, y) < \delta_x$, then $|g_{N_x}(x) - g_{N_x}(y)| < \varepsilon/2$. By the triangle inequality and $f_n(x) \geq f_{n+1}(x)$, we conclude that

$$\forall x \exists N_x \in N, \delta_x > 0 \forall n > N_x, d(x, y) < \delta_x : g_n(y) < \varepsilon.$$

   The open balls $B_{\delta_x}(x)$ are cover of X because $\{x\} \subset B_{\delta_x}(x)$, so $X = \bigcup_{x\in X} \{x\} \subset \bigcup_{x\in X} B_{\delta_x}(x)$. Because X is compact, there exist a finite subcover $B_{\delta_{x_k}}(x)$, $k = 1, \ldots, m$. Set $N = \max\{n_{x_1}, \ldots, n_{x_m}\}$. Let $n > N$ and $y \in X$. Then there exists k such that $y \in B_{\delta_{x_k}}(x)$, thus $N \geq N_{x_k}$, consequently $g_n(y) < \varepsilon$. We have proved that $g_n \rightrightarrows 0$ on X, thus $f_n \rightrightarrows f$ on X.

5. Let $L < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable with the property that $\sup_{x\in\mathbb{R}} f'(x) < L$. Prove that there exists a fixed point for f, i.e., that there exists $x \in \mathbb{R}$ such that $f(x) = x$. Hint: Consider the function $g(x) = x - f(x)$.

   Solution.

   (a) Define $g(x) = x - f(x)$. Then $g'(x) = 1 - f'(x) \geq 1 - L > 0$. From the mean value theorem, if , then

$$x > 0 \Rightarrow g(x) - g(0) = g'(\xi)\, x \geq (1 - L)\, x$$

   and

$$x < 0 \Rightarrow g(x) - g(0) = g'(\xi)\, x \leq (1 - L)\, x$$

   Consequently,

$$x > 0 \Rightarrow g(x) \geq g(0) + (1 - L)\, x$$

   and

$$x < 0 \Rightarrow g(x) \leq g(0) + (1 - L)\, x$$

   Thus, there exists points a such that $g(a) \leq 0$ and $b > a$ such that $g(b) \geq 0$, so by the intermediate value theorem, there exists x such that $g(x) = 0$, that is, $f(x) = x$.

6. Define $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that derivatives of f of all orders exist at 0, and $f^{(n)}(0) = 0$.

   Solution. First we prove by induction for all $n = 0, 1, \ldots$ and $x \neq 0$,

$$f^{(n)}(x) = \frac{p_n(x)}{q_n(x)} e^{-\frac{1}{x^2}},$$

where p_n and q_n are some polynomials. Clearly, $p_0 = q_0 = 1$. Suppose the statement hods for some $n \geq 0$. Then

$$\begin{aligned}
f^{(n+1)}(x) &= \left(\frac{p_n(x)}{q_n(x)} e^{-\frac{1}{x^2}} \right)' \\
&= \frac{p_n'(x) q_n(x) - q_n'(x) p_n(x)}{q_n^2(x)} e^{-\frac{1}{x^2}} + \frac{p_n(x)}{q_n(x)} \frac{-2}{x^3} e^{-\frac{1}{x^2}} \\
&= \left(\frac{p_n'(x) q_n(x) - q_n'(x) p_n(x)}{q_n^2(x)} + \frac{p_n(x)}{q_n(x)} \frac{-2}{x^3} \right) e^{-\frac{1}{x^2}}
\end{aligned}$$

where the bracket is again a rational function. Now we show by induction that $f^{(n)}(0)$ exists and equals 0. For $n = 0$, this is by definition. Suppose that we already know that for some n. By the definition of derivative,

$$\begin{aligned}
f^{(n+1)}(0) &= \lim_{x \to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x \to 0} \frac{1}{x} f^{(n)}(x) \\
&= \lim_{x \to 0} \frac{1}{x} \frac{p_n(x)}{q_n(x)} e^{-\frac{1}{x^2}} = 0
\end{aligned}$$

since $\lim_{y \to \infty} y^k e^{-y} = 0$ for any k, we have $\lim_{y \to \infty} R(y) e^{-y^2} = 0$ for any rational function R, and so

$$\lim_{x \to 0} \frac{p(x)}{q(x)} e^{-\frac{1}{x^2}} = 0$$

for any polynomials p, q.

7. Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ Riemann integrable. Prove that for all $\epsilon > 0$ there exists $g \in \mathcal{C}([a, b])$ such that

$$\int_a^b |f - g| \, dx < \epsilon.$$

Hint: Define g piecewise.

Solution. Since f is Riemann integrable, there exists a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that $U(f, P) - L(f, P) < \varepsilon$ where

$$L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}), \quad U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1})$$

and

$$m_i = \inf \{f(x) : x \in [x_{i-1}, x_i]\}, \quad M_i = \sup \{f(x) : x \in [x_{i-1}, x_i]\}.$$

Note that from this definition

$$\begin{aligned}
m_1 &\leq f(x_0) \leq M_1 \\
\max\{m_i, m_{i+1}\} &\leq f(x_i) \leq \min\{M_i, M_{i+1}\}, i = 1, \dots, n-1 \\
m_n &\leq f(x_n) \leq M_n
\end{aligned}$$

Now define g as the picewise linear function given by the values $f(x_i) = g(x_i)$, $i = 0, \ldots, n$. Then

$$m_i \leq g(x) \leq M_i, \quad x \in [x_{i-1}, x_i], i = 1, \ldots, n$$

and since also

$$m_i \leq f(x) \leq M_i, \quad x \in [x_{i-1}, x_i], i = 1, \ldots, n$$

we have

$$|f(x) - g(x)| \leq M_i - m_i, \quad x \in [x_{i-1}, x_i], i = 1, \ldots, n$$

Consequently

$$\int_a^b |f - g| \, dx \leq \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) = U(f, P) - L(f, P) < \varepsilon$$