PHD PRELIMINARY EXAMINATION IN APPLIED ANALYSIS
JANUARY 19, 2018

Name: ________________________________

- The examination consists of 6 problems. Do problems 1 to 4, and two out of 5, 6, and 7. If you submit all three, only 5 and 6 will be graded.
- Each problem is worth 20 points. Numbered parts of a problem have equal weight.
- Justify your solutions: cite theorems that you use, provide counter-examples, give explanations.
- Write legibly using a dark pencil or pen. Rewrite your solution if it gets too messy.
- Begin solution to every problem on a new page; write only on one side of a sheet; number all pages throughout; just in case, write your name on every page.
- Do not submit scratch paper.
- Ask the proctor if you have any questions.

Good luck!

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

Total ________

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(1) Let (X, d_X) and (Y, d_Y) be metric spaces and $K \subset X$ be compact. Prove the standard result that if $f : X \to Y$ is continuous, then $f(K) \subset Y$ is compact.

(2) Let (X, d) be a metric space, $A \subset X$ nonempty, and for any $x \in X$, define the distance from x to A as

$$\operatorname{dist}(x, A) = \inf\{d(x, a) \,:\, a \in A\}.$$

We say that y is a limit point of A if for all $r > 0$ there exists $a \in A$ such that $d(a, y) < r$. Using this definition of a limit point, show that y is a limit point of A if and only if $\operatorname{dist}(y, A) = 0$.

(3) Let $a < b$ be real numbers and (f_n) be a sequence of contraction maps such that $f_n : [a, b] \to [a, b]$ for all n, prove the following:

(a) There exists a uniformly convergent subsequence (f_{n_k}).

(b) If f denotes the limit of the uniformly convergent subsequence, then there exists $x \in [a, b]$ such that $f(x) = x$.

(4) Let (X, d) be a compact metric space and (f_n) a sequence of continuous real-valued functions defined on X that converge pointwise to a continuous function f. Prove the standard result: if $f_n(x) \geq f_{n+1}(x)$ for all $x \in X$ and $n \in \mathbb{N}$, then (f_n) converge uniformly.

(5) Let $L < 1$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable with the property that $\sup_{x \in \mathbb{R}} f'(x) < L$. Prove that there exists a fixed point for f, i.e., that there exists $x \in \mathbb{R}$ such that $f(x) = x$. Hint: Consider the function $g(x) = x - f(x)$.

(6) Define $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Prove that derivatives of f of all orders exist at 0, and $f^{(n)}(0) = 0$.

(7) Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ Riemann integrable. Prove that for all $\varepsilon > 0$ there exists $g \in \mathcal{C}([a, b])$ such that

$$\int_a^b |f - g| \, dx < \varepsilon.$$

Hint: Define g piecewise.